AD-A270 138

MTL Massachusetts Technological Laboratory, Inc.
312 Austin Street, West Newton, MA 02165 (617) 965-3992

Signal Compression for C^3 Applications Using Hyperdistributions

Final Report for Contract:
#DAAL01-91-C-0013
15 January 1991 – 15 July 1991

Prepared for:

U.S. Army Laboratory Command
ET&D Laboratory SLCET-I
Fort Monmouth, NJ 07703-5000

Prepared by:

Dr. Ta-Ming Fang
Massachusetts Technological Laboratory, Inc.
312 Austin Street, W. Newton, MA 02165 (617) 484-2296

Best
Available
Copy

MTL Massachusetts Technological Laboratory, Inc.
312 Austin Street, West Newton, MA 02165 (617) 965-3992

Signal Compression for C^3 Applications Using Hyperdistributions

Final Report for Contract:
#DAAL01-91-C-0013
15 January 1991 – 15 July 1991

Prepared for:

U.S. Army Laboratory Command
ET&D Laboratory SLCET-I
Fort Monmouth, NJ 07703-5000

Prepared by:

Dr. Ta-Ming Fang
Massachusetts Technological Laboratory, Inc.
312 Austin Street, W. Newton, MA 02165 (617) 484-2296

Abstract

We develop our previous work on hyperdistributions into a formally well-defined transform which may be applied to images, the hyperdistribution transform (HDT). The HDT has many properties in common with conventional orthogonal transforms of signals, such as the Fast Fourier Transform, which suggests the possiblity of developing a fast algorithm for the HDT. Presently, we have formulated the HDT in matrix language, which permits a reasonably efficient computational approach to calculating the HDT of an image. We then apply the HDT to image compression by representing the image as a truncated HDT expansion and reconstructing the image from the truncated HDT expansion. The compression ratio is measured in terms of the number of bits in the truncated HDT expansion compared to the number of bits in the original image. Test cases involving both synthetic and natural images are considered. Good quality reconstructions of natural images are obtained with compression ratios of 4:1 and recognizable images are obtained with compression ratios of 16:1. It was not necessary to segment the images into sub-images. Substantial further improvements in the performance of HDT compression may be obtained by employing image segmentation and other standard techniques for transform-based image compression algorithms.

Contents

1 **Introduction** 4

2 **Derivation of Hyperdistribution Transforms** 6

3 **Discrete Formulation of Transforms** 8

4 **Matrix Formulation of Transforms** 9

5 **Implementation Scheme** 10

6 **Results of Compression/Decompressions** 11

7 **Conclusions** 29

A **Computer Code** 30

B **Preprints of Publications** 36

1 Introduction

Even modest-sized images involve large amounts of data. A 512×512 image in which each pixel may take on 256 gray scale values will require 1/4 Mbyte of computer memory for storage. (1 Mbyte = 1,048,576 = 2^{20} bytes) Accordingly, storage of large numbers of images places large demands on computer memories. Transmission of images puts correspondingly large demands on channel bandwidth and/or time required for transmission of images. For these reasons, among others, ther are considerable economic and practical pressures to develop representations of images which require smaller numbers of bits.

Fundamentally, compression algorithms exploit the redundancy (in an information theoretic sense) of the pixel intensities in an image. This redundancy is reflected, for example, in the statistical dependence of these pixel values. A compressed image will require fewer bits than the original image. Generally speaking, image compression techniques fall into two categories[1], those which depend on coding techniques, such as Huffman coding, and those which depend on transform techniques, such as the discrete cosine transform (DCT) and the hyperdistribution-based methods which we have developed as the subject of this proposed research program. We shall concentrate attention on the class of transform-based compression algorithms.

Exact reconstruction of an image may be obtained by lossless algorithms, usually at some cost in the compression ratio achievable (typically ~ 10). Applications which can tolerate approximate reconstructions of images may utilize lossy algorithms for compression/decompression. These lossy algorithms may be particularly appropriate when the least significant bits (LSB's) of an image are corrupted by noise and thus do not represent desirable data. Lossy compression/decompression algorithms can achieve higher compression ratios (40 or more).

A number of techniques have been developed for image compression/decompression, such as the DCT and techniques based on affine transformations and fractals. Such techniques are computationally intensive and, particularly for the case of the DCT, may require implementation on special hardware to achieve compression/decompression of images in reasonable times. We note that the marketplace for commercial applications in image compression is,

in fact, moving in the direction of special purpose chips for rapid execution of particular algorithms.

A compression/decompression algorithm which may be used in conjunction with other image processing algorithms, such as spatial filtering, edge-enhancement, or deconvolution techniques may have special advantages. In our Phase I research we have developed such an approach based on newly developed mathematical called hyperdistributions. Hyperdistributions are theoretically attractive because they are an algebraic field in which convolutions are the multiplicative operation. Thus, problems for which the calculation of the convolution inverse (deconvolution) is not well-posed by conventional Fourier transform techniques, may be solved uniquely with hyperdistribution techniques. This property also is reflected in the comparative computational stability and efficiency of deconvolution computations by hyperdistribution techniques.

On the basis of connections with the problem of moments and formal analogies with moment expansions, we believed that hyperdistribution expansion techniques would have a significant utility for problems in image compression. In practice, this would mean that an image would be represented in a two-dimensional hyperdistribution expansion which would be truncated at a finite number of terms. The image could then be reconstructed to some level of precision from its hyperdistribution expansion, in a fashion analogous to that we have demonstrated for using hyperdistributions for computing global approximations of functions. We expected that for many images, the number of bits required for the hyperdistribution expansion would be significantly smaller than the number of bits in the original image, i.e. that the hyperdistribution representation of an image is a valid compressed representation of the image.

Our Phase I research effort developed the mathematical structure required for effectively carrying out these algorithms, and then conclusively demonstrated the validity of this point of view. We include in this proposal some of the first demonstrations of image compression/decompression using hyperdistributions. We are able to achieve compression ratios of 16:1 routinely even with algorithms at the present crude stage of development. We believe that by fine-tuning these algorithms and combining them with other conventional image processing algorithms, that higher compression ratios and improved fidelity of the reconstructed

image is possible. Our proposed Phase II effort will lead to hyperdistribution algorithms for image compression/decompression competitive with other entries in the commercial arena.

We establish an explicit method for image compression and reconstruction using hyperdistribution theory. The point function approximation for the hyperdistribution expansion, the Rodriguez hyperdistribution expansion, is used to formulate a hyperdistribution transform (HDT). The transform includes an adjustable parameter which is used to vary the shapes of the reconstructing wave and optimize the reconstruction performance. A matrix representation of the transform similar to other methods is derived. This establishes algorithms for carrying out HDT's analogous to those for conventional orthogonal transforms such as the FFT. This method is tested on three images demonstrating various compression ratios currently attainable.

2 Derivation of Hyperdistribution Transforms

The hyperdistribution expansion is formulated as a transform which is similar to other orthogonal transform methods with the addition of an adjustable parameter which controls the shapes of the basis functions. Begin with the truncated Rodriguez hyperdistribution expansion in two dimensions as derived previously. The expansion for the approximation to a function $f(x,y)$ is defined as

$$\hat{f}(x,y) = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} a_{nm}^{\lambda} \left[(-1)^n H_n^{\lambda}(x) \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda}\right] \left[(-1)^m H_m^{\lambda}(y) \frac{e^{-y^2/\lambda^2}}{\sqrt{\pi}\lambda}\right] \tag{1}$$

with coefficients

$$a_{nm}^{\lambda} = \frac{(-1)^{n+m}\lambda^{2(n+m)}}{2^{n+m}\, n!\, m!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H_n^{\lambda}(x) H_m^{\lambda}(y) f(x,y)\, dx\, dy. \tag{2}$$

Substitute the definition for scaled Hermite polynomials, $H_n^{\lambda}(x) = H_n(x/\lambda)/\lambda^n$, to get the expansion

$$\hat{f}(x,y) = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} a_{nm}^{\lambda} \left[\frac{(-1)^n H_n(x/\lambda)}{\lambda^n} \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda}\right] \left[\frac{(-1)^m H_m(y/\lambda)}{\lambda^m} \frac{e^{-y^2/\lambda^2}}{\sqrt{\pi}\lambda}\right] \tag{3}$$

with coefficients

$$a_{nm}^{\lambda} = \frac{(-1)^{n+m}\lambda^{n+m}}{2^{n+m}\, n!\, m!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H_n(x/\lambda) H_m(y/\lambda) f(x,y)\, dx\, dy. \tag{4}$$

Insert equation (4) into (3); cancel $(-1)^{n+m}\lambda^{n+m}$, separate $1/(2^{n+m}\, n!m!)$ into square roots, and combine $1/\lambda$ with each differential. Then separate the equation again to get the expansion

$$\hat{f}(x,y) = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} a_{nm}^{\lambda} \frac{H_n(x/\lambda) H_m(y/\lambda)}{\sqrt{2^{n+m}\, n!\, m!}} \frac{e^{-(x^2+y^2)/\lambda^2}}{\pi} \tag{5}$$

with coefficients

$$a_{nm}^{\lambda} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{H_n(x/\lambda) H_m(y/\lambda)}{\sqrt{2^{n+m}\, n!\, m!}} f(x,y) \frac{dx}{\lambda} \frac{dy}{\lambda}. \tag{6}$$

Now substitute for the function $f(x,y)$

$$f(x,y) = g(x,y) \left[\frac{e^{-(x^2+y^2)/\lambda^2}}{\pi}\right]^{1-(1/\alpha)} \tag{7}$$

to get the expansion

$$\hat{g}(x,y) = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} a_{nm}^{\lambda} \frac{H_n(x/\lambda)}{\sqrt{2^n n!}} \frac{H_m(y/\lambda)}{\sqrt{2^m m!}} \left[\frac{e^{-(x^2+y^2)/\lambda^2}}{\pi}\right]^{1/\alpha} \tag{8}$$

with coefficients

$$a_{nm}^{\lambda} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{H_n(x/\lambda)}{\sqrt{2^n n!}} \frac{H_m(y/\lambda)}{\sqrt{2^m m!}} \left[\frac{e^{-(x^2+y^2)/\lambda^2}}{\pi}\right]^{1-(1/\alpha)} g(x,y) \frac{dx}{\lambda} \frac{dy}{\lambda}. \tag{9}$$

The parameter α has been introduced into the formulation which is used to vary the shape of the wavelets that are used to determine the approximate function (i.e. reconstruct the image). The shape of the wavelets will affect the rate and type of convergence of the expansion series. In previous Hyperdistribution formulation, α can be considered equal to one. In the case were α is equal to two, the wavelets form orthonormal bases. This case is explored in the following work

Notice that $1/\lambda$ appears with x and y every where except in the function. The variable substitution

$$\begin{aligned} x &= \lambda u \\ y &= \lambda v \end{aligned} \tag{10}$$

$$\begin{aligned} dx &= \lambda\, du \\ dy &= \lambda\, dv \end{aligned} \tag{11}$$

produces the hyperdistribution parametric wavelet transforms with the expansion

$$\hat{g}(\lambda u, \lambda v) = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} a_{nm} \frac{H_n(u)}{\sqrt{2^n n!}} \frac{H_m(v)}{\sqrt{2^m m!}} \left[\frac{e^{-(u^2+v^2)}}{\pi} \right]^{1/\alpha} \tag{12}$$

with coefficients

$$a_{nm}^{\lambda} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{H_n(u)}{\sqrt{2^n n!}} \frac{H_m(v)}{\sqrt{2^m m!}} \left[\frac{e^{-(u^2+v^2)}}{\pi} \right]^{1-(1/\alpha)} g(\lambda u, \lambda v)\, du\, dv. \tag{13}$$

3 Discrete Formulation of Transforms

The equations (12) and (13) deal with the expansion for a continous function, but we are interested in discretely sampled functions. An actual image is given as a matrix of values which can be considered discrete samples of a function. Consider a $p \times p$ sampled image where i,j, are integers and s_i,s_j, define a symmetric coordinate system for the image. define

$$\begin{aligned} u &= s_i \quad i = 1 \ldots p \\ v &= s_j \quad j = 1 \ldots p \end{aligned} \tag{14}$$

and the function t in terms of these integer values

$$t_{ij} = g(\lambda u, \lambda v) = g(x, y). \tag{15}$$

From equations (12) and (13) with $\alpha = 2$ the expansion for approximation to t using orthonormal wavelets is

$$\hat{t}_{ij} = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} \tau_{nm} \left[\frac{H_n(s_i)}{\sqrt{2^n n!}} \left(\frac{e^{-s_i^2}}{\sqrt{\pi}} \right)^{1/2} \right] \left[\frac{H_m(s_j)}{\sqrt{2^m m!}} \left(\frac{e^{-s_j^2}}{\sqrt{\pi}} \right)^{1/2} \right] \tag{16}$$

with coefficients $\tau_{nm} = a_{nm}^{\lambda}$ using the discrete version of equation (13)

$$\tau_{nm} = \sum_{i=1}^{p}\sum_{j=1}^{p} t_{ij} \left[\frac{H_n(s_i)}{\sqrt{2^n n!}}\left(\frac{e^{-s_i^2}}{\sqrt{\pi}}\right)^{1/2}\right] ds \left[\frac{H_m(s_j)}{\sqrt{2^m m!}}\left(\frac{e^{-s_j^2}}{\sqrt{\pi}}\right)^{1/2}\right] ds. \tag{17}$$

We move ds to make equations (16) and (17) symmetric and get the expansion

$$\hat{t}_{ij} = \sum_{n=0}^{k-1}\sum_{m=0}^{k-1} \tau_{nm} \left[\frac{H_n(s_i)}{\sqrt{2^n n!}}\left(\frac{e^{-s_i^2}ds}{\sqrt{\pi}}\right)^{1/2}\right]\left[\frac{H_m(s_j)}{\sqrt{2^m m!}}\left(\frac{e^{-s_j^2}ds}{\sqrt{\pi}}\right)^{1/2}\right] \tag{18}$$

with coefficients

$$\tau_{nm} = \sum_{i=1}^{p}\sum_{j=1}^{p} t_{ij} \left[\frac{H_n(s_i)}{\sqrt{2^n n!}}\left(\frac{e^{-s_i^2}ds}{\sqrt{\pi}}\right)^{1/2}\right]\left[\frac{H_m(s_j)}{\sqrt{2^m m!}}\left(\frac{e^{-s_j^2}ds}{\sqrt{\pi}}\right)^{1/2}\right]. \tag{19}$$

4 Matrix Formulation of Transforms

The discretized formulation can be represented in more conventional form by using a matrix representation similar to other orthogonal transforms. Equations (18) and (19) suggest using matrix operations for doing the calculations.

Consider the image $p \times p$ matrix

$$\mathbf{I} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1p} \\ t_{21} & t_{22} & \cdots & t_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t_{p1} & t_{p2} & \cdots & t_{pp} \end{pmatrix} \tag{20}$$

If we define a $k \times p$ Hermite matrix

$$\mathbf{H} = \begin{pmatrix} \frac{H_0(s_1)}{\sqrt{2^0 0!}}\left(\frac{e^{-s_1^2}ds}{\sqrt{\pi}}\right)^{1/2} & \frac{H_0(s_2)}{\sqrt{2^0 0!}}\left(\frac{e^{-s_2^2}ds}{\sqrt{\pi}}\right)^{1/2} & \cdots & \frac{H_0(s_p)}{\sqrt{2^0 0!}}\left(\frac{e^{-s_p^2}ds}{\sqrt{\pi}}\right)^{1/2} \\ \frac{H_1(s_1)}{\sqrt{2^1 1!}}\left(\frac{e^{-s_1^2}ds}{\sqrt{\pi}}\right)^{1/2} & \frac{H_1(s_2)}{\sqrt{2^1 1!}}\left(\frac{e^{-s_2^2}ds}{\sqrt{\pi}}\right)^{1/2} & \cdots & \frac{H_1(s_p)}{\sqrt{2^1 1!}}\left(\frac{e^{-s_p^2}ds}{\sqrt{\pi}}\right)^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{H_{k-1}(s_1)}{\sqrt{2^{k-1}(k-1)!}}\left(\frac{e^{-s_1^2}ds}{\sqrt{\pi}}\right)^{1/2} & \frac{H_{k-1}(s_2)}{\sqrt{2^{k-1}(k-1)!}}\left(\frac{e^{-s_2^2}ds}{\sqrt{\pi}}\right)^{1/2} & \cdots & \frac{H_{k-1}(s_p)}{\sqrt{2^{k-1}(k-1)!}}\left(\frac{e^{-s_p^2}ds}{\sqrt{\pi}}\right)^{1/2} \end{pmatrix} \tag{21}$$

The HD coefficients, τ_{nm}, can now be calculated by viewing equation (19) as the following operation

$$\mathbf{A} = \mathbf{H} \cdot \mathbf{I} \cdot [\mathbf{H}]^{\mathbf{T}} \tag{22}$$

which creates a $k \times k$ HD coefficient matrix

$$\mathbf{A} = \begin{pmatrix} \tau_{00} & \tau_{01} & \cdots & \tau_{0(k-1)} \\ \tau_{10} & \tau_{11} & \cdots & \tau_{0(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{(k-1)0} & \tau_{(k-1)1} & \cdots & \tau_{(k-1)(k-1)} \end{pmatrix} \tag{23}$$

We can reconstruct the image by viewing equation (18) as the following operation

$$\hat{\mathbf{I}} = [\mathbf{H}]^{\mathbf{T}} \cdot \mathbf{A} \cdot \mathbf{H} \tag{24}$$

which creates a $p \times p$ reconstructed image matrix

$$\hat{\mathbf{I}} = \begin{pmatrix} \hat{t}_{11} & \hat{t}_{12} & \cdots & \hat{t}_{1p} \\ \hat{t}_{21} & \hat{t}_{22} & \cdots & \hat{t}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{t}_{p1} & \hat{t}_{p2} & \cdots & \hat{t}_{pp} \end{pmatrix} \tag{25}$$

5 Implementation Scheme

An algorithm for implementing the previous formulas to images is developed in this section. The image is given as a $p \times p$ matrix of one byte integers.

A coordinate system is determined for the image through the definitions

$$s_i = -L + (i-1)\, ds \tag{26}$$

$$ds = \frac{2\,L}{p-1} \tag{27}$$

where we consider

$$\mathbf{I}_{ij} \equiv \mathbf{I}(s_i, s_j) \tag{28}$$

The remaining variables to be determined are the length of the coordinate space of the image, $(2 \times L)$, and the dimensions of the HD coefficient matrix, $(k \times k)$. This gives us the parameters L and k to adjust.

We predict that the optimal value for L for reconstruction will be approximately equal to the zero closest to infinity of the highest order wavelet used in the reconstruction. This criterion insures that all the wavelets completely fit inside the coordinate space of the image and that wavelets can have sign variations at the edge of the coordinate space of the image. This establishes L as a function of k. Results of reconstruction with various values for L substantiate this function as good preliminary criterion.

The zeros of the nth wavelet are equivalent to the zeros of the nth order Hermite polynomial, since this is the only part of the equation for the wavelet that can change sign. For a particular choice of k we determine L by finding the first zero from infinity of the $(k-1)$th order Hermite polynomial.

The compression ratio is defined

$$\mathrm{CR} = \frac{\text{number of bits in the image matrix}}{\text{number of bits in HD coefficient matrix}} = \frac{\mathrm{p \times p\,(8bits)}}{\mathrm{k \times k\,(32bits)}} = \frac{\mathrm{p}^2}{4\,\mathrm{k}^2} \tag{29}$$

The mean squared error is defined

$$\mathrm{MSE} = \frac{1}{\mathrm{p \times p}} \sum_{\mathrm{i=1}}^{\mathrm{p}} \sum_{\mathrm{j=1}}^{\mathrm{p}} (\mathbf{I}_{\mathrm{ij}} - \hat{\mathbf{I}}_{\mathrm{ij}})^2 \tag{30}$$

6 Results of Compression/Decompressions

The first image analyzed is a 64×64 artificial terrain called "Island". This image was created by combining three gaussian functions of various widths and strengths. Gaussian signals are a natural test case for reconstruction. The original image is shown by surface and contour plots in figure (1). The image reconstructed from a 32×32 HD coefficient matrix is shown in figure (2). The image reconstructed from a 16×16 HD coefficient matrix is shown in figure (3).

The second image analyzed is a 256×256 image called "Gordon". The original image is shown in figure (4). The reconstructed images representing compression ratios of 1, 4, and 16 are shown in figures (5), (6), and (7).

To demonstrate the effects of the parameter L we analyze the reconstruction of "Gordon" with higher and lower then predicted values for L. The compression ratio for each reconstruction is 4. The predicted value for L was used in figure(6). Figure (8) shows a reconstruction with L one half the predicted value. Figure (9) shows a reconstruction with L one and a half the predicted value.

The third image analyzed is a 512×512 image called "Liberty". The original image is shown in figure (10). The reconstructed images representing compression ratios of 1, 4, and 16 are shown in figures (11) and (12).

The results obtained from these three images should be considered in the nature of an existence proof. We have demonstrated that compressions of image data can be achieved with HDT's and recognizable results obtained for the reconstructed images. It is particularly remarkable that these results have been obtained for unsegmented images. Additional work to be undertaken during the Phase II research effort will improve the compression ratio and the fidelity of reconstruction. For example, conventional techniques such as image segmentation and image processing, combined with optimization of the HD wavelet parameters, can be confidently predicted to yield continued improvements in the performance of the HDT compression algorithm.

Figure Captions

- **Figure 1** Original Image "Island" of 64×64 pixels; (a) surface plot (b) contour plot.
- **Figure 2** Reconstructed Image "Island" from 32×32 HD coefficient matrix with CR = 1 and MSE = 0.00451 . (a) surface plot (b) contour plot.
- **Figure 3** Reconstructed Image "Island" from 16×16 HD coefficient matrix with CR = 4 and MSE = 0.232 . (a) surface plot (b) contour plot.

- **Figure 4** Original Image "Gordon" of 256×256 pixels.
- **Figure 5** Reconstructed Image "Gordon" with CR $= 1$ and MSE $= 13.6$.
- **Figure 6** Reconstructed Image "Gordon" with CR $= 4$ and MSE $= 25.7$.
- **Figure 7** Reconstructed Image "Gordon" with CR $= 16$ and MSE $= 59.7$.
- **Figure 8** Reconstructed Image "Gordon" using low L with CR $= 4$ and MSE $= 61.8$.
- **Figure 9** Reconstructed Image "Gordon" using high L with CR $= 4$ and MSE $= 1104$.
- **Figure 10** Original Image "Liberty" of 512×512 pixels.
- **Figure 11** Reconstructed Image "Liberty" with CR $= 4$ and MSE $= 227$.
- **Figure 12** Reconstructed Image "Liberty" with CR $= 16$ and MSE $= 476$.

Figure 1 Original Image "Island" of 64 × 64 pixels; (a) surface plot (b) contour plot.

Figure 1a

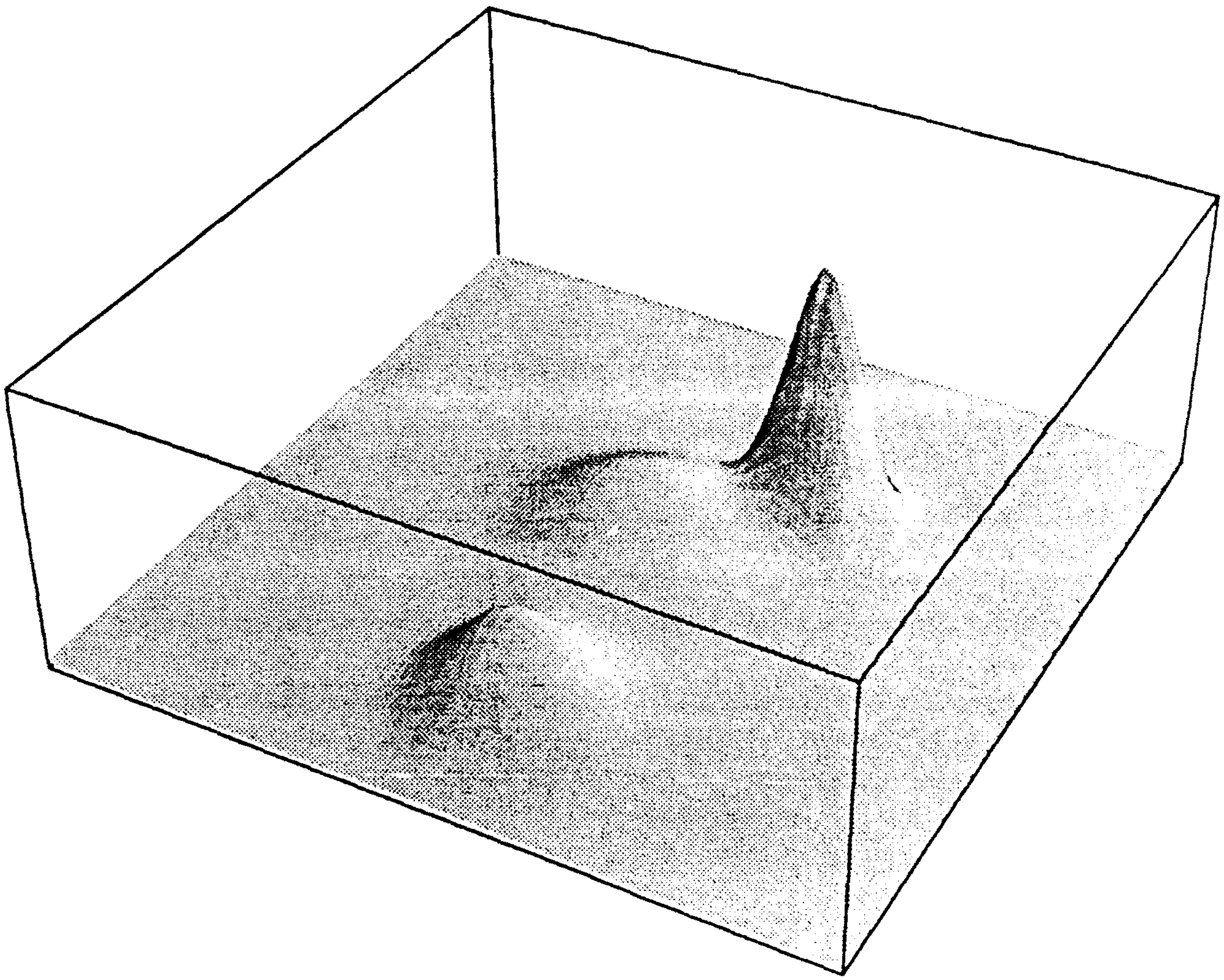


Figure 1b

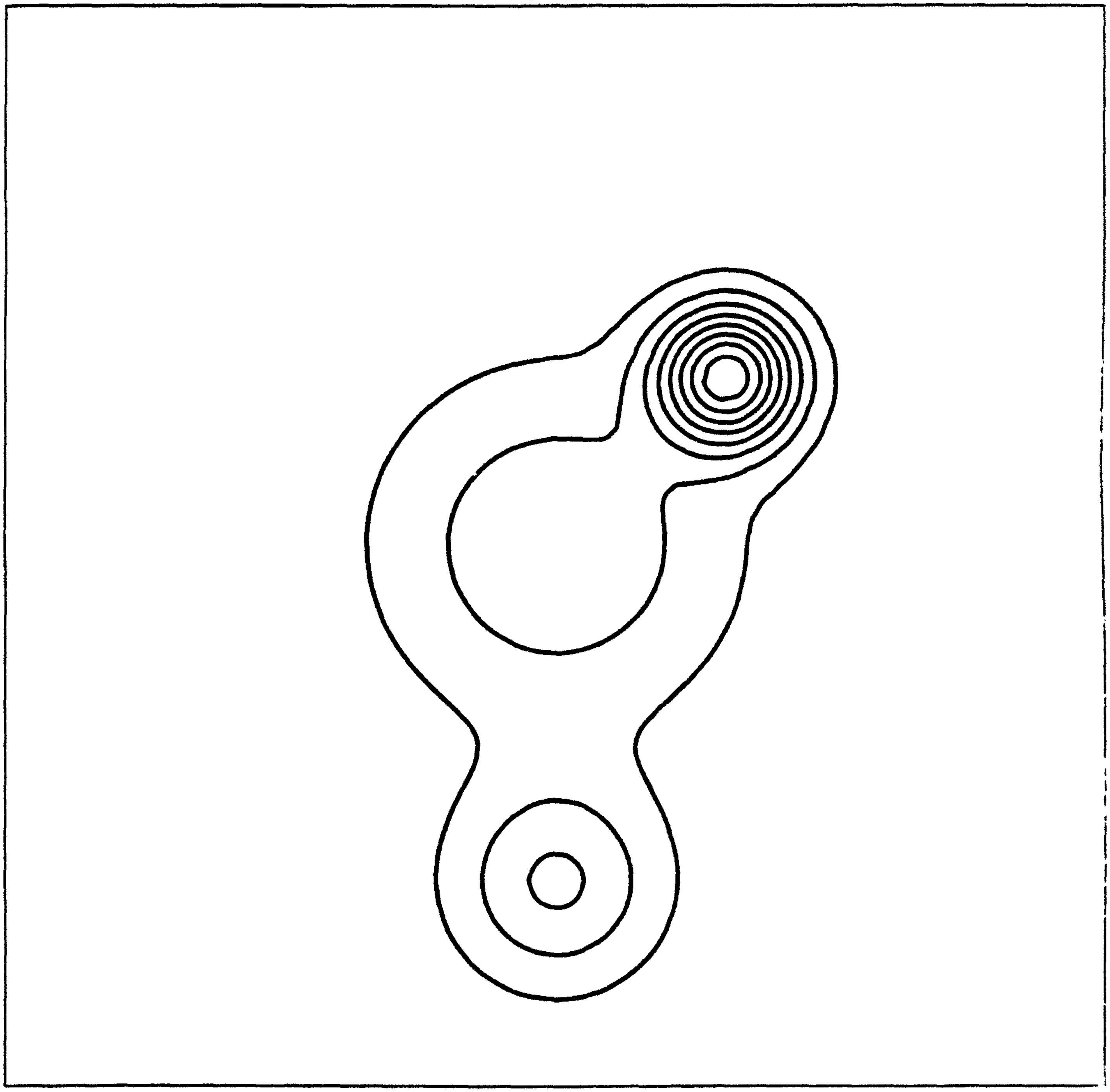


Figure 2 Reconstructed Image "Island" from 32×32 HD coefficient matrix with CR 1 and MSE **= 0.00451 . (a) surface plot (b) contour plot.**

Figure 2a

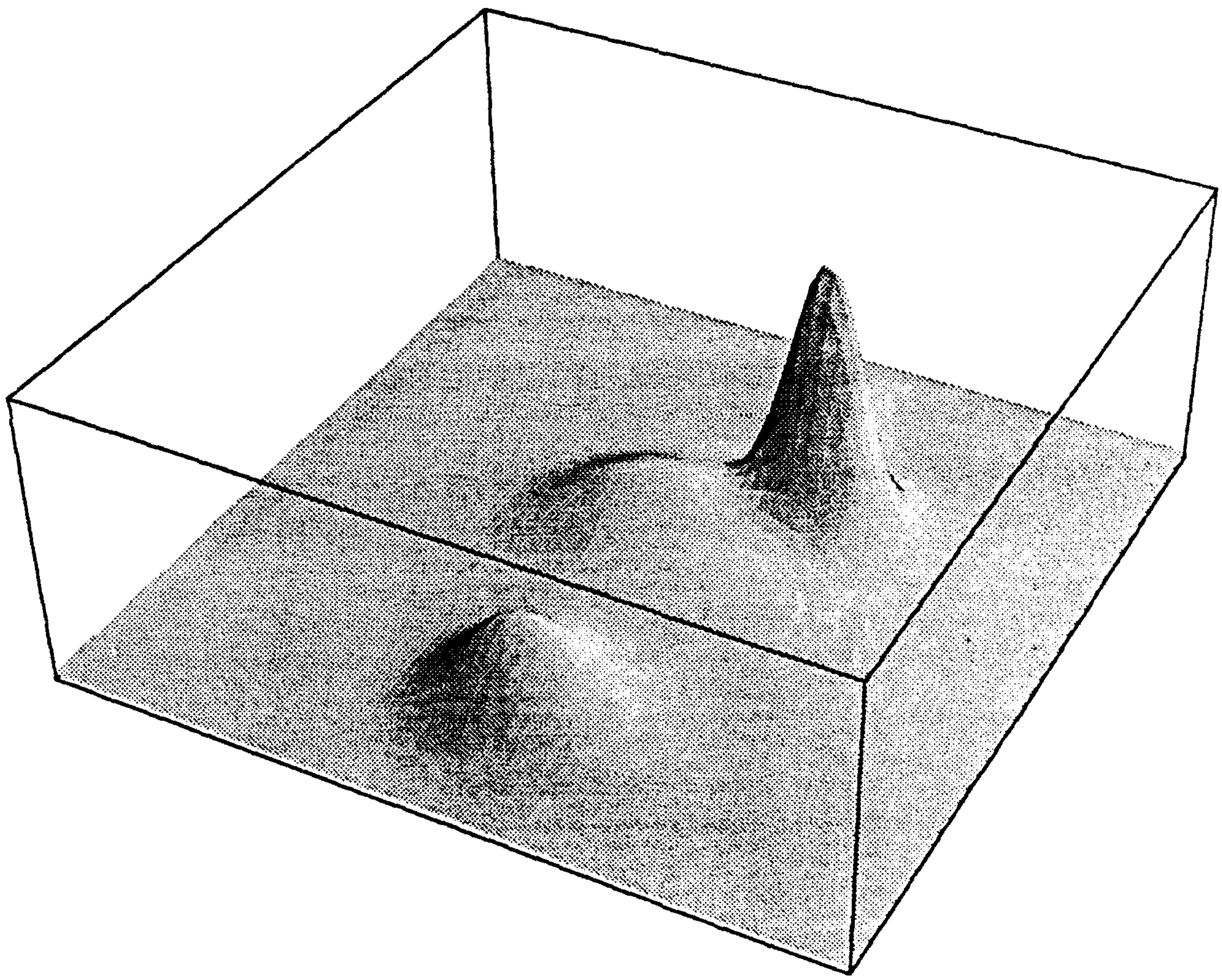


Figure 2b

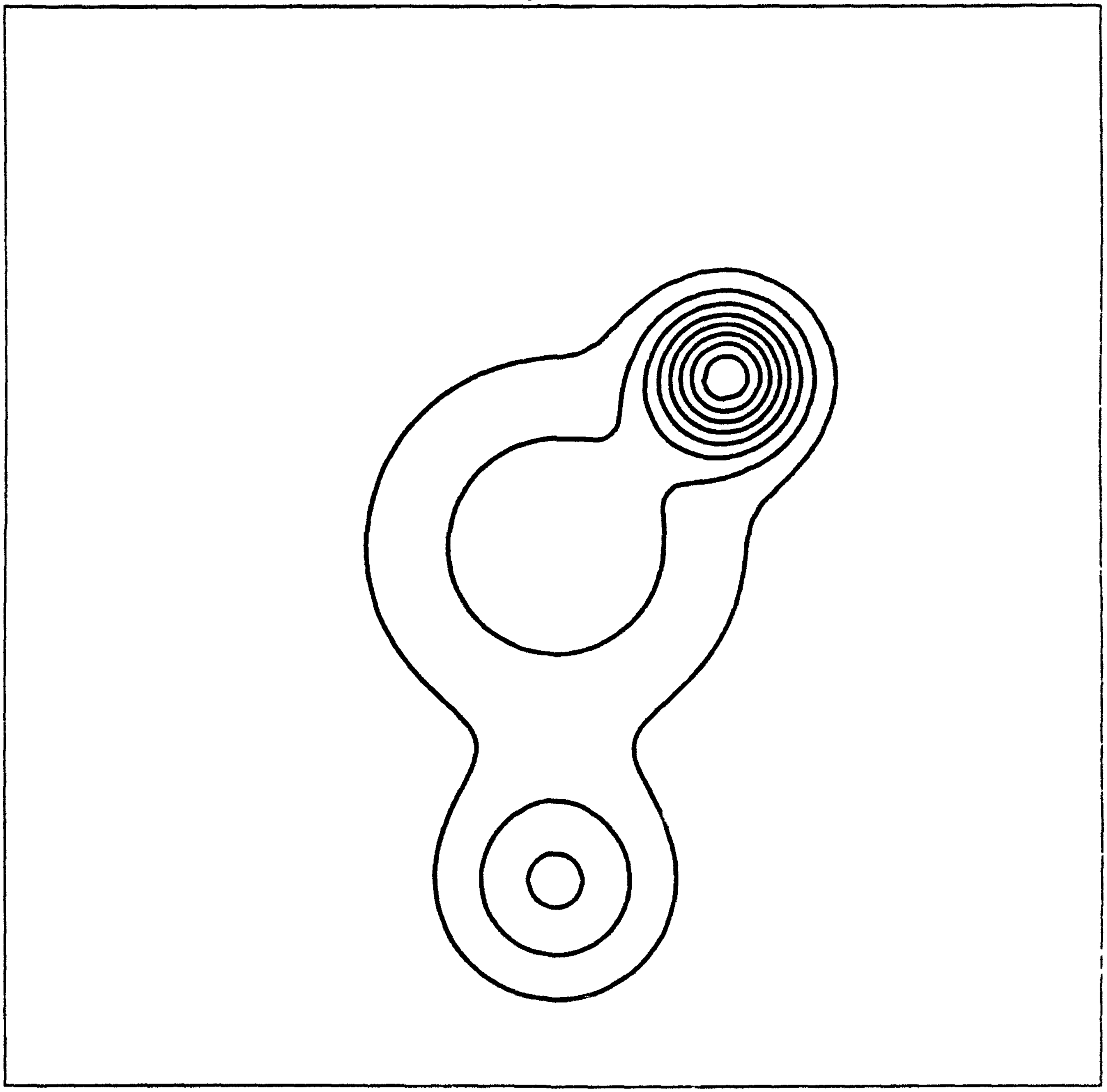


Figure 3 Reconstructed Image "Island" from 16×16 HD coefficient matrix with CR 4 and MSE = 0.232 . (a) surface plot (b) contour plot.

Figure 3a

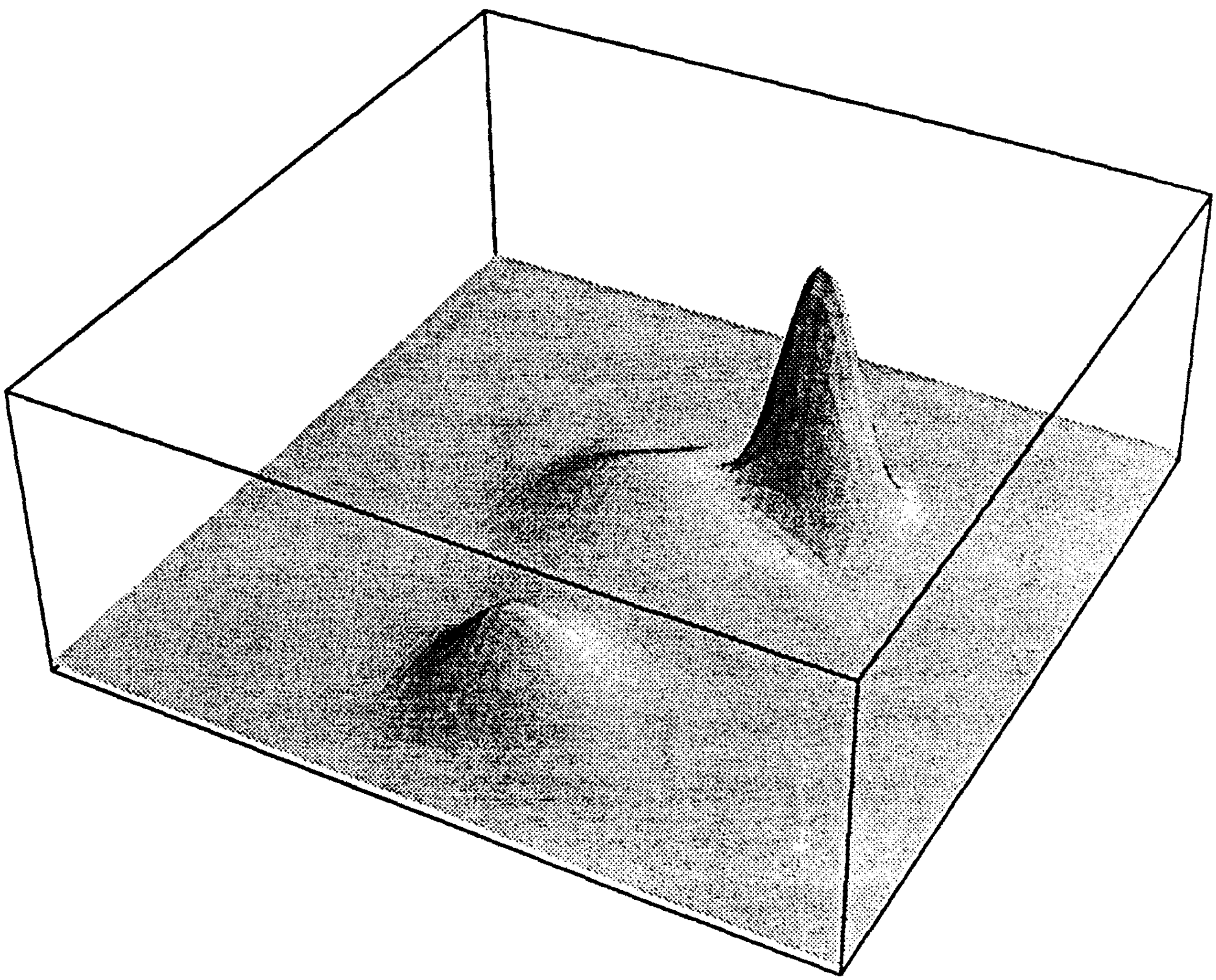


Figure 3b

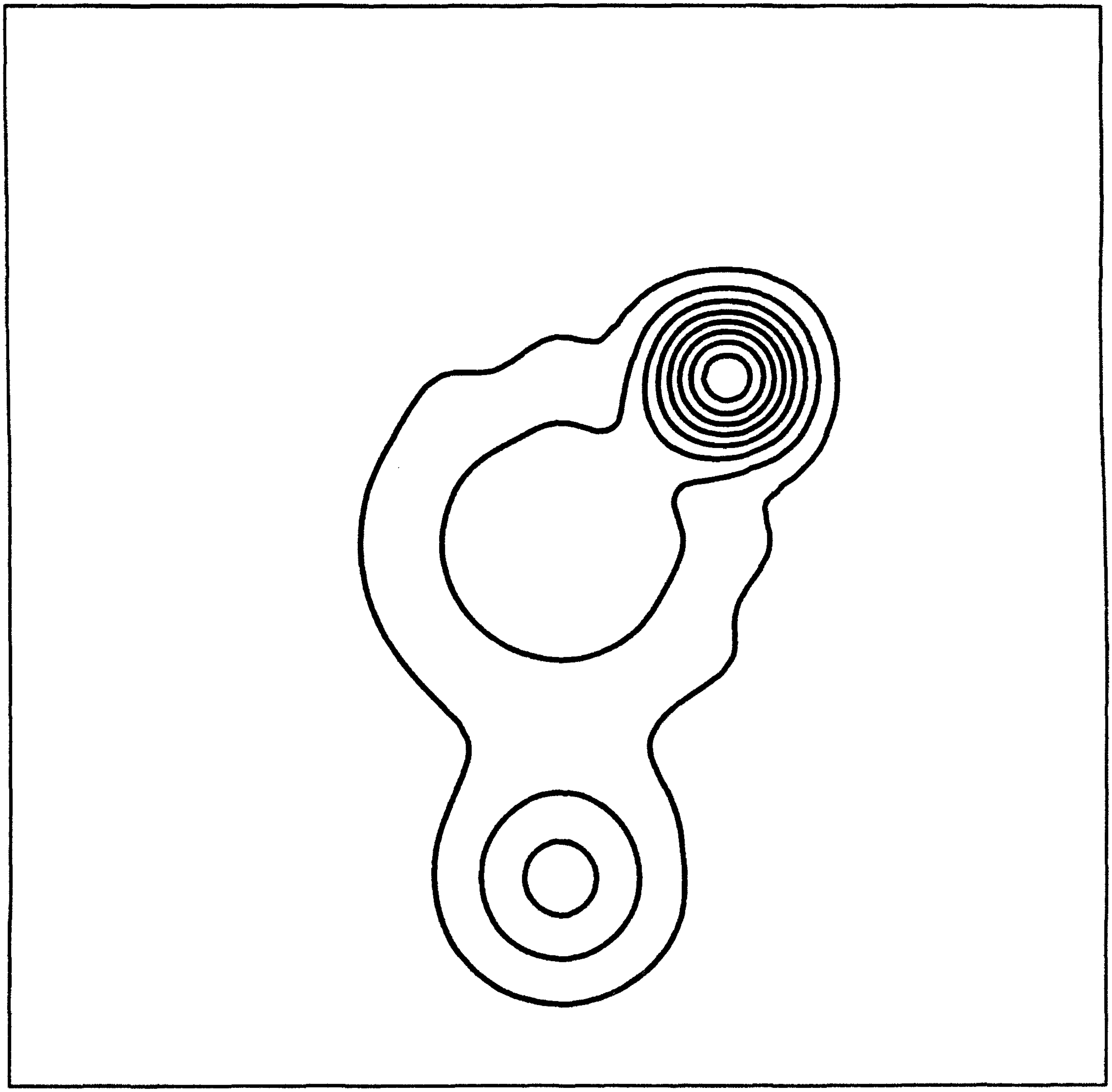


Figure 4

Figure 6

Figure 7

Figure 6

Figure 9

Figure 10

Figure 11

Figure 12

7 Conclusions

We have established a transform description of the hyperdistribution approach to image processing, the hyperdistribution transform (HDT). The properties of the HDT have been outlined in a fashion analogous to more conventional transforms, such as the Fast Fourier Transform. We have demonstrated compression of both synthetic and natural images using a truncated HDT expansion. For unsegmented natural images, compression ratios of 4:1 and 16:1 were demonstrated. Standard techniques used for conventional image compression algorithms should allow further improvements in the performance of HDT image compression.

A Computer Code

The computer program used to perform the analysis is included here. The language used is standard ANSI C. The utility routine used are from **Numerical Recipes** *Press et. al.* and are not included in this report.

```
/************************************************************************
 * This program is used to calculate the Hyperdisdribution transfrom *
 * of an image and to reconstruct the approximate to the image from  *
 * the Hyperdistribution transfrom.                                   *
 ***                                                                  *
 * by Thomas W. Drueding                date : 3/9/91                 *
 *     Aero-Mech Dept.                                                *
 *     Boston University                                              *
 *     (617) 353-5260                                                 *
 ************************************************************************/

#include"malloc.h"
#include"nrutil.h"
#include<stdio.h>
#include<math.h>

#define PI 3.14159265359

void image_read(),image_write(),init_grid(),make_hdimg(),make_rlimg();
```

```
main(argc,argv)
int argc;
char *argv[];
{
  float **image,**recon,**hdimg,**trans,limit;
  int p,k;

  /* Inititialize paramters */
  if (argc < 3) {
    printf("Usage: hd (p) (k) (limit) \n");
    exit();
  }
  sscanf(argv[1],"%d",&p);   /* x/y size of real image (assume square image) */
  sscanf(argv[2],"%d",&k);   /* x/y size of HD   image (assume square image) */
  sscanf(argv[3],"%f",&limit);   /* edge of coordinate system on the image   */

  /* Allocate dynamic memory for matrices */
  image = matrix(1,p,1,p);       /* image : original image                     */
  recon = matrix(1,p,1,p);       /* recon : reconstructed image                */
  hdimg = matrix(0,k-1,0,k-1);   /* hdimg : HD image (HD transform of image) */
  trans = matrix(0,k-1,1,p);     /* trans : HD transformation matrix           */

  /* PROCESS */
  image_read(image,p,"image.b");           /* read original image from file     */
  init_grid(trans,k,p,limit);              /* setup HD Transformation matrix    */
  make_hdimg(image,hdimg,trans,k,p);       /* calculate HD transform of image   */
  make_rlimg(recon,hdimg,trans,k,p);       /* calculate image from HD transfrom */
  image_write(recon,p,"recon.b");          /* write reconstructed image to file */
}
```

```
/* Read BINARY image from file */
void
  image_read(image, size, filename)
float **image;
int size;
char *filename;
{
  FILE *fp;
  int i,j;

  if ((fp = fopen(filename, "rb")) == NULL){
    printf("Cannot open file %s\n", filename);
    exit(0);
  }
  printf("\nReading %s...\n", filename);
  for (i=1; i<=size; ++i)
    for (j=1; j<=size; ++j)
      image[i][j] = ((float) getc(fp)) - 127.0;
  fclose(fp);
  printf("...done\n");
}
```

```
/* Write BINARY image to file */
void
  image_write(image,  size, filename)
float **image;
int size;
char *filename;
{
  FILE *fp;
  int i,j;
  float val;

  if ((fp = fopen(filename, "wb")) == NULL){
    printf("Cannot open file %s\n", filename);
    exit(0);
  }
  printf("\nWriting %s...\n", filename);
  for (i=1; i<=size; ++i)
    for (j=1; j<=size; ++j){
      val = image[i][j] + 127.0;
      if (val < 0.0) val = 0.0;  /* check bounds */
      else if (val > 255.0) val = 255.0;
      putc( (char) val , fp);   /* convert to binary and write */
    }
  fclose(fp);
  printf("...done\n");
}
```

```
/* setup HD Transformation matrix    */
void init_grid(trans,k,p,limit)
float **trans,limit;
int k,p;
{
  int n,i;
  float s,ds,Const,Tn2,Tn1,Tn0;

  printf("\nInitializing grid ...\n");
  ds = 2.0 * limit / (p-1.0);
  Const = sqrt( ds / sqrt(PI) );
  for (i=1; i<=p; i++){
    s = -limit + (i-1) * ds;
    Tn1 = 0.0;
    trans[0][i] = Tn0 = exp(-s * s/2) * Const;
    for (n=1; n<=k-1; n++ ){
      Tn2 = Tn1;
      Tn1 = Tn0;
      trans[n][i] = Tn0 = s * sqrt((float)2/n)* Tn1 - sqrt((float)(n-1)/n) * Tn2;
    }
  }
  printf("...done\n");
}
```

```
/* calculate HD transform of image   */
void make_hdimg(rlimg,hdimg,trans,k,p)
float **rlimg,**hdimg,**trans;
int k,p;
{
  int i,j,n,m;
  printf("\nMaking HD image...\n");
  for (n=0; n<=k-1; n++)
    for (m=0; m<=k-1; m++)
      for (i=1; i<=p; i++)
for (j=1; j<=p; j++)
   hdimg[n][m]+=rlimg[i][j]*trans[n][i]*trans[m][j];
  printf("...done\n");
}

/* calculate image from HD transfrom */
void make_rlimg(rlimg,hdimg,trans,k,p)
float **rlimg,**hdimg,**trans;
int k,p;
{
  int i,j,n,m;
  printf("\nMaking RL image...\n");
  for (i=1; i<=p; i++)
    for (j=1; j<=p; j++)
      for (n=0; n<=k-1; n++)
for (m=0; m<=k-1; m++)
   rlimg[i][j]+=hdimg[n][m]*trans[n][i]*trans[m][j];
  printf("...done\n");
}
```

B Preprints of Publications

In this appendix we attach two preprints of publications submitted under this contract.

Hyperdistributions I: One Dimensional Analysis

T. Drueding[1], T. Fang[2], R. Hohlfeld[1], W. Ma[1], G. Sandri[1]

[1] Boston University College of Engineering
110 Cummingtion Street, Boston, MA 02215

[2] Massachusetts Technological Laboratory, Inc.
330 Pleasant Street, Belmont, MA 02178

Submitted for Publication: SIAM Journal for Numerical Analysis

Hyperdistributions I: One Dimensional Analysis

We develop in this paper and the following paper a new technique for the calculation of convolution products and their inverses. This is accomplished by constructing a class of singular "functions", hyperdistribution, that form a closed algebraic field with the convolution product as the multiplicative operation. In this paper we consider functions of a single variable. This one-dimensional algebra can be applied to signal processing. Furthermore, in this paper we use the construction of hyperdistributions to obtain a novel parametric approximation method. We demonstrate the use of our approximation method with simple examples.

Introduction

In the sequel, we develop a novel tool for the calculation of convolution products and their inverses: hyperdistributions which give simple applications to the two related areas of signal analysis and image processing.

Hyperdistributions are singular "functions", defined and constructed below, that we find to form a algebraic abelian field with the convolution product as the multiplicative operation.

Signal analysis is a natural application of hyperdistribution in one (one dimensional) variable. Image processing is a corresponding application for hyperdistributions in a two dimensional variable. Tomography corresponds to three dimensional variable and the budding field of space-time processing corresponds to four dimensions.

The outline of the paper is as follows. In section I we introduce hyperdistributions heuristically, we discuss the convolution group and derive a remainder theorem. In section II we construct rigorously hyperdistributions by a modified Hermite polynomial expansion and we use the tools of the Christoffel-Darboux theory to obtain sufficient conditions for L^2 convergence. In section III we show that Gauss' multipole expansion is obtained explicitly as a simple application of the hyperdistribution inverse. Finally, in section IV we expand a gaussian function in terms of derivatives of a different gaussian to demonstrate the use of our new parametric expansion and the concurrent minimization of error. The logical interconnection of the sections is shown in Fig 1.

We note that, in effect we introduce a method for establishing and approximating solutions of integral equations of convolution form based on the use of this new class of singular functions. We demonstrate with examples that there are cases for which our method is applicable, but Fourier transform methods fail. Our method requires the calculation of the moments of the given functions and of the kernel rather than calculation of their Fourier coefficients. This property motivates consideration of examples for which our method is clearly preferable to Fourier transform techniques. Applications are given with emphasis on image deconvolution and the analysis of turbulent flows.

1 Heuristic definition of Hyperdistribution.

We introduce a general approximation whose integral properties are the focus of interest. Our approximation displays in configuration space the properties of the classical moment generating expansion for the Fourier transform of the probability distribution. Moments, and even shapes, are shown to be captured well by our expansion. In addition, our expansion allows consideration of "functions" which are more singular than temperate (i.e. Fourier-analyzable) distributions, but that can be represented by infinite sequences of distributions. We call these "functions" generalised distributions. We show that generalised distributions form the appropriate framework for carrying out the process of deconvolution, in fact, we give a straightforward algorithm, the "Bochner-Martin algebra", to compute explicitly the convolution inverse of any generalised distribution. Applications of the method to simple optical deconvolutions are shown to be straightforward. These applications have been motivated originally by our studies of optics and turbulent flows. We have since become aware of the much wider applicability of our methods.

1.1 Taylor and moment expansions

The theory of generalised distributions is built on ideas related to Dirac's delta function, which is technically a "generalised function". The Dirac delta function, $\delta(x)$, has the properties

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1 \tag{1}$$

and

$$f(x) = \int_{-\infty}^{+\infty} \delta(x - x')f(x')dx' \tag{2}$$

for any suitably smooth function, $f(x)$. We term equation(2), "Dirac's identity".

The Dirac delta function is symmetric in its argument, i.e. $\delta(x - x') = \delta(x' - x)$ and, since the Dirac delta function is a generalised function , we may Taylor expand $\delta(x' - x)$ about x',

$$\delta(x' - x) = \delta(x') - x\delta'(x') + x^2\delta''(x') + \ldots + R^n(x) \tag{3}$$

Here $\delta'(x) = d\delta(x)/dx$, $\delta''(x) = d^2\delta(x)/dx^2$, etc. This expression and similar ones below are assumed to hold under integration. Equation (3) allows one to compute a "local" approximation to $f(x)$, since if we substitute this expansion into Dirac 's identity, we recover the usual Taylor expansion of $f(x)$ about $x = 0$,

$$f(x) = f(0) + xf'(0) + x^2/2!f''(0) + \ldots \tag{4}$$

This approximation is local in the sense that it requires derivatives of $f(x)$ at a single point, $x = 0$, and in general has a limited radius of convergence. On the other hand, if we expand $\delta(x - x')$ about x, we have

$$\delta(x - x') = \delta(x) - x'\delta'(x) + x'^2/2!\delta''(x) + \ldots \tag{5}$$

When this series is substituted into Dirac 's identity, we obtain

$$f(x) = M^0\delta(x) - M^1\delta'(x) + M^2/2!\delta''(x) + \ldots + R^n(x) \tag{6}$$

The coefficients M^n, defined by

$$M^n = \int_{-\infty}^{+\infty} x^n f(x)dx \tag{7}$$

are simply the moments of the function $f(x)$. Therefore equation (6) is an approximation of $f(x)$ involving global information about $f(x)$, that is, the moments of the function.

This then may be taken as a motivation for our definition of a generalised distribution as a function which may be written in the form,

$$f(x) = \sum_{n=0}^{\infty} a_n\delta^n(x) \tag{8}$$

The a_n values are coefficients given by

$$a_n = (-1)^n M^n/n! \tag{9}$$

Note that equation (8) is equivalent to the familiar moment generating expansion of probability theory (see equation (6)).

1.2 The Convolution Group

If we have two generalised distributions f_1 and f_2, their linear combination $\lambda f_1 + \mu f_2$ is also a generalised distribution (where λ and μ are real coefficients). The pth derivative of a generalised distribution, $d^r f(x)/dx^r = \nabla^r f(x)$, is a generalised distribution. Also, the convolution of two generalised distribution $f_1 * f_2$ is a generalised distribution. These may all be thought of as "closure properties" of generalised distributions.

Generalised distributions allow us to make an effective computation of the convolution inverse. Given a generalised distribution f, the desired convolution inverse $In[f]$ satifies

$$f * In[f] = \delta \tag{10}$$

Here δ represents Dirac 's delta function, which is the identity of the convolution operation. We shall show by our construction that $In[f]$ is a generalised distribution. Writing the convolution explicitly,

$$f * In[f] = \int_{-\infty}^{\infty} dx' f(x') In[f](x - x') = \delta(x) \tag{11}$$

It is now necessary to compute the product of the sums and match coefficients. Taking, $f(x) = \sum_n a_r \nabla^n (x)$, and $In[f(x)] = \sum_n b_n \nabla^n \delta(x)$, we see that the computation of the convolution inverse is effectively the determination of a collection of b_n values, given a set of a_n values. Substituting into equation (11),

$$\int_{-\infty}^{+\infty} dx' (\sum_{p=0}^{\infty} a_p \nabla'^r \delta(x'))(\sum_{q=0}^{\infty} b_q \nabla^q \delta(x - x')) = \delta(x) \tag{12}$$

Noting that $\nabla_{x-x'} = \nabla_x$ and that $\nabla^r \delta(x) * \nabla^q \delta(x - x') = \nabla^{r+q} \delta(x)$, equation (12) may finally be reduced to

$$\sum_p (a_p \sum_q b_q \nabla^{p+q} \delta(x)) = \delta(x) \tag{13}$$

or equivalently with $r = p + q$

$$\sum_{p=0}^{\infty} \sum_{r=0}^{\infty} a_p b_{r-p} \nabla^r \delta(x) = \delta(x) \tag{14}$$

Matching coefficients on the left hand and right hand sides implies that only the $r = 0$ term survives. The result is a linear system of equations for the b values in terms of the a values. It is easiest to see the behavior by writing the first few equations in this linear system,

$$\begin{aligned} a_0 b_0 &= 1 \\ a_0 b_1 + a_1 b_0 &= 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0 \\ a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 &= 0 \end{aligned} \tag{15}$$

and so forth. Thus, we can see that, $b_0 = 1/a_0$, $b_1 = -a_1/a_0^2$ $b_2 = a_1^2/a_0^3 - a_2/a_0^2$, etc. This completes the computation of the terms of the convolution inverse.

1.3 Fourier Transforms and simple examples

Lastly, we shall want to consider the Fourier transform of a generalised distribution. Again taking $f(x) = \sum_{n=0}^{\infty} a_n \bigtriangledown^n \delta(x)$, we can immediately evaluate the Fourier transform as:

$$\begin{aligned} \int_{-infty}^{+\infty} e^{ikx} f(x) dx &= \sum_{n=0}^{\infty} a_n \int_{-\infty}^{+\infty} e^{ikx} \bigtriangledown^n \delta(x) dx \\ &= \sum_{n=0}^{\infty} a_n (-1)^n \int_{-\infty}^{+\infty} (\bigtriangledown^n e^{ikx}) \delta(x) dx \\ &= \sum_{n=0}^{\infty} a_n (-i)^n k^n \end{aligned} \tag{16}$$

where for the last step we have used $\bigtriangledown^n e^{ikx} = i^n k$. The Fourier transform of a generalised distribution may be seen to be a power series.

The Fourier transform of a function is the "moment generating function" because of equation (9). Thus the basic requirement for the validity of writing a function as a generalised distribution is that its moments be finite, or more stringently, that its Fourier transform be real analytic.

Generalisation of this discussion to a two-dimensional generalised distribution, as would be required for most types of imaging data, is relatively straightforward. This generalisation retains the mathematical properties,

particularly for convolutions and Fourier transforms, that make generalised distributions attractive for signal processing.

Simple explicit examples of convolution inverses are obtained from the standard Green 's functions. The one-dimensional Helmholtz equation, for example, can be treated equally easily with generalised distributions and with Fourier transforms to obtain both the Green 's functions and its convolution inverse. Thus, we have

$$(d^2/dx^2 + 1/4)e^{-|x|/2} = -\delta(x) \tag{17}$$

Application of equation (15) shows that only b_0 and b_2 contribute to the convolution inverse and, in fact, writing $\hat{f} = Inv[f]$,

$$Inv[e^{-|x|/2}] = 1/2\delta(x) - \delta''(x) \tag{18}$$

This result is easily reproduced by using Fourier transforms and is easily verified using equation (10).

In contrast, the convolution inverse of the Gaussian function cannot be obtained as a Fourier transform, but it is easily obtained using generalised distributions. It may be verified that for a Gaussian function of width σ, the convolution inverse is readily obtained from equation (15), while calculation using Fourier transforms leads to a divergent result. Calculation of moments and use of equation (15) yields

$$\begin{aligned} a_{2n} &= (\sigma/\sqrt{2})^n/n! \\ b_{2n} &= (-1)^n(\sigma/\sqrt{2})^n/n! \end{aligned} \tag{19}$$

2 Rigorous Construction of Hyperdistributions

In this section we first modify the classical expansion in the Hermite polymials. This expansion is "superior" to a power series expansion in that the terms are orthogonal, making the error orthogonal to the approximation. Our modification maintains this advantage. Furthurmore, the standard Christoffel-Darboux analysis gives a very useful sufficient condition for convergence that our expansion inherits from the Hermite polymial expansion. Since our expansion utilises the Rodriguez formula for the Hermite polymials, we call it the "Rodriguez expansion".

2.1 Modified Hermite Expansion: Rodriguez expansion

In effect, we discuss here a systematic pointwise approximation of generalised distributions. This approximation method is analogous to the method developed by Temple to approximate distributions by smooth functions. The Temple method has become widely known through Lighthill 's monograph: Fourier Series and Generalised Functions.

Generalised distributions are approximate by a modification of the classical expansion in Hermite polymials (hereafter called the "Hermite expansion") which is suggested by the Rodriguez formula:

$$(-1)^n H_n(x) e^{-x^2} = \nabla^n e^{-x^2} \tag{20}$$

where ∇ denotes the derivative. For simplicity, we consider here the one-dimensional problem, so that ∇ is the ordinary derivative with respect to the variable x. To see how the Hermite expansion can be transformed into an approximation for generalised distributions, consider a function $f(x)$ which we want to represent as a generalised distribution. Multipy $f(x)$ by

$$\frac{e^{+x^2/\lambda^2}}{\sqrt{\pi}\lambda} \tag{21}$$

then expand the resulting expression in terms of the "scaled Hermite polymials

$$H_n^\lambda(x) = \lambda^{-n} H_n(x/\lambda) \tag{22}$$

whose definition is justified by the formula (25) below. We obtain the expression

$$f(x)\frac{e^{x^2/\lambda^2}}{\sqrt{\pi}\lambda} = \sum_{n=0}^{\infty} \alpha_n^\lambda H_n^\lambda(x) \tag{23}$$

Multiply both sides of the equation by the normalised Gaussian introduced by Temple, i.e.

$$\delta_\lambda(x) = \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda} \tag{24}$$

And, observe that by rescaling the Rodriguez formula (20) we can write

$$(-1)^n H_n^\lambda(x)\delta_\lambda(x) = \nabla^n \delta_\lambda(x) \tag{25}$$

The final result is

$$f(x) = \sum_{n=0}^{\infty} C_n^\lambda \, \nabla^n \, \delta_\lambda(x) \tag{26}$$

where we have introduced the coefficients

$$\begin{aligned} C_n^\lambda &= (-1)^n \alpha_n^\lambda \\ \alpha_n^\lambda &= \frac{\lambda^{2n}}{2^n n!}\int_{-\infty}^{+\infty} f(x) H_n^\lambda(x) dx \end{aligned} \tag{27}$$

When we let λ tend to zero, we have a representation of $f(x)$ as a generalised distribution. We call the expansion (26) the "Rodriguez expansion for $f(x)$ with width λ". The width parameter is a novel feature of the Rodriguez expansion when compared to standard expansions in complete sets of basis functions. The standard expansions do not contain a free parameter. The advantages of the Rodriguez expansion will be demonstrated below in the context of the theory of generalised distributions and of convolution inverses.

We consider as an example a function familiar from the analysis of turbulence spectra, the "Ogura" function

$$f(x) = N e^{-x^4} \tag{28}$$

With N determined by the normalisation condition,

$$\int_{-\infty}^{+\infty} f(x) dx = 1 \tag{29}$$

we have

$$N = (1/2)\Gamma(1/4) \approx 1.8128 \tag{30}$$

A number of graphs are now used to illustrate the three main advantages of the Rodriguez expansion.

1. The Rodriguez expansion converges pointwise. (See Fig 3.) The Hermite expansion also converges pointwise, but the convergence is slow for large values of the argument. In addition, the Hermite expansion exhibits " whipping tails" for large values of the argument.(See Fig 6.)

2. The pointwise convergence of the Rodriguez expansion is independent of λ, but the rate of convergence can be optimised by a proper choice of λ. (See Fig 3)

3. In addition to pointwise (i.e. local) convergence, the Rodriguez expansion exihibits a global convergence manifested by accurate representation of moments of the function being approximated. To illustrate this global convergence, since the function we are considering here is symmetric in x, we have found it convenient to consider "one-sided moment" plots to exihibit the convergence of our generalised distribution approximations. These are integrals over positive x only, i.e. $\mu_n(x) = \int_0^x (x')^n f(x')dx'$. The one-sided moments of order n are given with excellent accuracy by numerical integration of the approximation with $n + 1$ terms. (See Figs 4 and 5.) Again in sharp contrast, the Hermite expansion gives divergent moments. (See Fig 7) Note that the horizontal axis in the one-sided moment plots is the argument x.

To summarise, the Hermite expansion exhibits pointwise convergence, albeit with "whipping tails". By contrast, the Rodriguez expansion exhibits pointwise convergence without "whipping tails". As a consequence, the Rodriguez expansion also yields a global approximation by accurately representing the moments. Furthurmore, the Rodriguez approximations has an adjustable parameter which may be selected to optimise the rate of convergence. The Christoffel-Darboux theory gives the necessary condition for convergence in appropriate L^2. For the Hermite expansion we have

$$\int_{-\infty}^{+\infty} e^{-x^2} f^2 dx < \infty \tag{31}$$

Inserting the transformation that leads to the Rodriguez expansion, we find

$$\int_{-\infty}^{+\infty} e^{+x^2} f^2 dx < \infty \tag{32}$$

for the corresponding L^2 definition for the Rodriguez series.

2.2 Definition of Hyperdistributions

In order to describe the process of antidiffusion, we have introduced a class of highly singular "functions", that are precisely defined in this section. Our process for defining hyperdistributions parallels the Temple definition (generalised function) as a good sequence of good functions. Good functions are smooth and tapered. More precisely, they are total point functions that are differentiable to all orders (C^∞), and decay at $\pm\infty$ faster than any power. Good functions play the role of "testing" a sequence of good functions for weak convergence. In fact, a sequence of good functions is a distribution if

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \phi(x) f_n(x) dx < \infty \tag{33}$$

for all good functions ϕ.

Since we conceive of hyperdistributions as "generalised" distributions, we are in fact implementing a second order generalisation of functions. Consequently, we need a double test as a convergence criterion. We implement this criterion by introducing *very good* functions $G_\Lambda(x)$ with the following properties:

1. $G_\Lambda(x)$ is smooth, that is, differentiable to all orders, C^∞.
2. $G_\Lambda(x)$ is essentially compact, i.e. it has a gaussian decay at $\pm\infty$: $G_\Lambda(x) \sim N e^{-x^2/\Lambda^2}$.

We will assume for convenience that G_Λ is normalised to unity:

$$\int_{-\infty}^{+\infty} G_\Lambda(x) dx = 1 \tag{34}$$

We define the width of G_Λ by

$$\Lambda^2/4 = \int_{-\infty}^{+\infty} (x - \bar{x})^2 G_\Lambda(x) dx \tag{35}$$

A primary example of very good functions is the gaussian, which we denote by $\delta_\lambda(x)$:

$$\delta_\lambda(x) = \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda} \tag{36}$$

We now introduce a sequence of very good functions defined by

$$\mathcal{H}_n^\lambda(x) = \sum_{k=0}^{n} a_k \nabla^k \delta_\lambda(x) \tag{37}$$

The sequence $\{\mathcal{H}_n^\lambda\}_{\lambda,n}$, where λ is a nonnegative real and n is a natural number, is a good sequence if, for all good functions ϕ and for all very good functions G_Λ, there exists a Λ_0 such that, for all $\Lambda > \Lambda_0$,

$$\lim_{\substack{n\to\infty \\ \lambda\to 0}} \int_{-\infty}^{+\infty} \phi(x)(\mathcal{H}_n^\lambda * G_\Lambda)(x)dx < \infty \tag{38}$$

We note that $e^{\pm t\nabla^2}\delta(x)$ are hyperdistributions. The sum (hyperdistribution)

$$\sum_{k=0}^{\infty} a_k \nabla^k \delta(x) \tag{39}$$

can thus be viewed either as a sequence of "good" distributions as $\lambda \to 0$:

$$\sum_{k=0}^{n} a_k \nabla^k \delta(x) \tag{40}$$

or, as $n \to \infty$, as sequence of good functions:

$$\sum_{k=0}^{\infty} a_k \nabla^k \delta_\lambda(x) \tag{41}$$

The latter representation is a *Rodriquez expansion*. The Rodriquez formula for the Hermite polymials can be used to show the derivatives of a gaussian form a complete set of orthogonal polynomials in an L^2 space. And thus the Rodriguez expansion yields a very useful point function approximation to any hyperdistribution:

$$\sum_{k=0}^{\infty} a_k(-1)^n \frac{H_n(x/\lambda)}{\lambda^n} \delta_\lambda(x) \tag{42}$$

where $H_n(x)$ denotes the Hermite Polynomial in x of order n.

3 The Multipole Expansion

We start with the familiar Poisson equation of potential theory,

$$\nabla^2 \phi = \rho \tag{43}$$

This equation is rewritten with the help of the (infinite domain) Green 's function

$$G(x) = \frac{-1}{4\pi r}, \qquad \nabla^2 = \delta \tag{44}$$

We can then rewrte the "potential", ϕ, in terms of the "charge distribution", ρ, as

$$\phi = G * \rho \tag{45}$$

Introduce Q with the property

$$Q * \rho = \delta \tag{46}$$

Convolving both sides of eq(45) with Q and using the commutative and associative properties of the * product, we find

$$\begin{aligned} Q * \phi &= Q * (G * \rho) \\ &= (Q * \rho) * G \\ &= G \end{aligned} \tag{47}$$

Solve eq(43) for ϕ in terms of the given G by computing the convolution inverse,

$$\phi(x) = \sum_{k=0}^{\infty} \lambda_k \nabla^k G(x) \tag{48}$$

which is Gauss' multipole series with coefficients

$$\lambda_k = \frac{(-1)^k}{k!} \int x^{\otimes n} \rho(x) d^3x \tag{49}$$

where $\otimes$ denotes tensor product and $x^{\otimes k}$ is the tensor power of the 3-vector x.

Substituting (49) into (48) we find a familiar expression

$$\phi(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\int y^{\otimes k} \rho(y) d^3y\right) \otimes \nabla^k \frac{-1}{4\pi r} \tag{50}$$

which is a standard result in potential theory.

We can interpret the Rodriguez expansion as a generalised multipole expansion that includes a size or radius parameter, λ. Thus our monopole generates the point source, which is a Dirac delta-function as a gaussian of width λ. We recover the standard multipoles when $\lambda \to 0$. Our expansion has the form

$$f(x) = a_0^\lambda \delta_\lambda(x) + a_1^\lambda \bigtriangledown \delta_\lambda(x) + a_2^\lambda \bigtriangledown^2 \delta_\lambda(x) + \ldots \tag{51}$$

The ingredients of the expansion are the basis functions

$$\delta_\lambda, \bigtriangledown\delta_\lambda, \bigtriangledown^2\delta_\lambda, \ldots \tag{52}$$

and the coefficients

$$a_0^\lambda, \qquad a_1^\lambda, \qquad a_2^\lambda, \ldots \tag{53}$$

all of which depend on the size parameter λ. The basis functions are, explicitly

$$\delta_\lambda(x) = \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda} \tag{54}$$

$$\bigtriangledown \delta_\lambda(x) = -2x\frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda} \tag{55}$$

$$\bigtriangledown^2 \delta_\lambda(x) = (4x^2/\lambda^2 - 2/\lambda^2)\frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda} \tag{56}$$

These functions are plotted in Figs 8,9 and 10 for 3 values of λ which show how the basis functions tend, as $\lambda \to 0$, to a monopole, a dipole and a quadrupole respectively. The corresponding expansion coefficients are given by

$$a_0^\lambda \xrightarrow{\lambda \to 0} \int_{-\infty}^{+\infty} f(x)dx \tag{57}$$

This coefficient represents the total area, or alternatively, the total mass or charge of the source function. Also

$$-a_1^\lambda \xrightarrow{\lambda \to 0} \int_{-\infty}^{+\infty} xf(x)dx \tag{58}$$

which represents the net dipole of the source. In mechanical terms a_λ^1 defines the center of mass of the source function. Finally

$$a_2^\lambda \xrightarrow{\lambda \to 0} \int_{\infty}^{+\infty} x^2 f(x)dx \tag{59}$$

which represents the quadrupole of the source distributions. In mechanical terms this is the moment of inertia of the mass distribution. Higher order basis function and coefficients have analogous interpretations.

4 Expansion of a Function in Rodriguez Series and its Optimization.

Our expansion preserves the classical properties that are derivable for orthogonal polynomial exapnsions (as opposed to power series), but also adds an important new feature: the size or radius parameter that generates the Gaussian picture of point monopole, dipole, quadrupole, etc, to a scenario in which we can allow for extended sources. We find from the convergence condition derived from the Christoffel-Darboux thoery in section II that convergence holds generally on a semi-infinite range of λ. This remarkable freedom is exploited in this section to optimise the rate of convergence of the expansion. This optimisation results in determining the value of λ for which a minimum number of terms is determin d in order to obtain a *given* tolerance. We measure the tolerance, as usual, by the least square fit integrated over the entire function. We define the error

$$\epsilon(N,\lambda) = \int_{-\infty}^{+\infty} dx(f(x) - \sum_{n=0}^{N} a_n^\lambda \nabla^n \delta_\lambda(x))^2 \tag{60}$$

We apply the formula to a standard gaussian, i.e. we take

$$f(x) = e^{-x^2}/\sqrt{\pi} \tag{61}$$

we then find, after some calculations

$$a_{2n}^\lambda = \sum_{k=0}^{n-1} \frac{(-1)^k 2^{k-2-n}(2n-2k-1)!!}{k!(2n-2k)!} + (-1)^n 2^{-2n}/n! \tag{62}$$

where the double factorial contains only odd terms. We can now express the standard gaussian in the Rodriguez form:

$$e^{-x^2}/\sqrt{\pi} = \sum_{n=0}^{\infty} \frac{(1-\lambda^2)^n}{2^{2n} n!} \nabla^{2n} \delta_\lambda(x) \tag{63}$$

A proof based on useful linear operator relations is as follows. We can check by differentiation in t and x that

$$\frac{e^{-x^2/4t}}{\sqrt{4\pi t}} = e^{t\nabla^2}\delta(x) \tag{64}$$

We now let

$$4t = \lambda^2 \tag{65}$$

As a consequence, we can write

$$\frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda} = e^{\lambda\nabla^2/4}\delta(x) = \delta_\lambda(x) \tag{66}$$

We now consider the identity

$$\begin{aligned} e^{\nabla^2/4}\delta(x) &= e^{\frac{1-\lambda^2}{4}\nabla^2} e^{\lambda^2\nabla^2/4}\delta(x) \\ &= e^{\frac{1-\lambda^2}{4}\nabla^2}(e^{\lambda^2\nabla^2/4}\delta(x)) \\ &= e^{\frac{1-\lambda^2}{4}\nabla^2}\delta_\lambda(x) \end{aligned} \tag{67}$$

expanding the R.H.S.

$$e^{\nabla^2/4}\delta(x) = e^{-x^2}/\sqrt{\pi} = \sum_{n=0} \frac{(1-\lambda^2)^n}{2^{2n}n!} \nabla^{2n}\,\delta_\lambda(x) \tag{68}$$

The method of proof will be used later. Furthurmore, it provides a welcome check on the rather difficult calculations of coefficients. We see by inspection that $\lambda = 1$ is optimal. Fig 11 shows the trend of the error function obtained numerically. Fig 12 shows the error function on an expanded scale. The behavior is smooth and gradient searches promise to be straightforward. We also consider the function

$$f(x) = cos(Kx)\frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda} \tag{69}$$

The overall behaviour is shown in Fig 13 and an enlargement in Fig 14. The function exhibits rapid variation. Fig 7 shows the choice of parameters $K = 4$ and $\lambda^2 = 3$. Fig 16 shows the behaviour of the error and the corresponding optimisation. In Fig 17 and 18 we show the result of deconvoluting the letter T with hyperdistribution algebra after it was "smeared" with a gaussian filter. In Fig 19, we simulate the crossberg-Todororich neural network for the cornsweet effect with hyperdistributions.

5 Conclusions

We have seen that generalised distribution provide a method for solving Fredholm integral equations of the convolution type which is competitive with the current methods that employ Fourier transforms. We have also seen that generalised distributions can be approximated numerically by sequences of smooth functions. This procedure is analogous to that of approximating Dirac δ functions by sequences of narrowing Gaussian functions. In this introductory paper we have used one-dimensional examples for illustrative purposes. We will discuss applications in higher dimensions and extensions of the theory of generalised distributions in following papers.

6 Appendix: Remainder Theorem for Hyperdistributions

In this appendix we give the remainder theorem for hyperdistributions. The remainder formula is of great use in calculating Lagrange and Cauchy estimates for the terms neglected. This feature of our expansion, like the presence of the adjustable "radius" parameter, is unique to our expansion and it is not shared by other orthogonal functions expansion.

The general form of the Taylor expansion for two distinct points y and z is:

$$F(y) = F(z) + (y - z) \bigtriangledown F(z) + \ldots + R^{n+1} \tag{70}$$

$$R^{n+1} = \int_z^y dt (y - t)^n \bigtriangledown^{n+1} F(t)/n! \tag{71}$$

The formula for the remainder is in fact an identity which is proven by recursion integrating by parts.

We now give the appropriate remainder for the two dual expansions discussed in section 1.

6.1 Taylor (local) expansion

We use

$$\begin{aligned} F(x) &= \delta(x - x') \\ &= \delta(x' - x) \\ y &= x' - x \\ z &= x' \end{aligned} \tag{72}$$

We have

$$\delta(x' - x) = \delta(x') - x \bigtriangledown \delta(x') + \ldots + \int_{x'}^{x' - x} dt (x' - x - t)^n \bigtriangledown^{n+1} \delta(t)/n! \tag{73}$$

Then we have, using Dirac 's identity

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \delta(x - x') f(x') dx' \\ &= f(0) - x \bigtriangledown f(0) + \ldots + R^{n+1} \end{aligned} \tag{74}$$

with the remarkable relation

$$
\begin{aligned}
R^{n+1} &= \int_{-\infty}^{+\infty} dx' f(x') \int_{x'}^{x'-x} dt(x' - x - t)^n \nabla^{n+1} \delta(t)/n! \\
&= \int_0^x dt(x - t)^n \nabla^{n+1} f(t)/n!
\end{aligned} \tag{75}
$$

6.2 Moment (global) expansion

We set

$$
\begin{aligned}
F(x) &= \delta(x - x') \\
y &= x - x' \\
z &= x
\end{aligned} \tag{76}
$$

We then have

$$
\delta(x - x') = \delta(x) - x'\delta(x) + \ldots + \int_x^{x-x'} dt(x - x' - t)^n \nabla \delta(t)/n! \tag{77}
$$

Using Dirac identity we conclude

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{+\infty} f(x')\delta(x - x')dx' \\
&= \delta(x) \int_{-\infty}^{+\infty} f(x')dx' - \nabla\delta(x) \int_{-\infty}^{+\infty} x' f(x')dx' + \ldots + R^{n+1} \\
R^{n+1} &= \int_{-\infty}^{+\infty} dx' f(x') \int_x^{x-x'} dt(x - x' - t)^n \nabla^{n+1} \delta(t)/n! \\
&= \int_{-\infty}^{+\infty} dx' f(x') \int_0^{x'} (-1)^n (y - x')^n \nabla^{n+1} \delta(x - y)dy/n!
\end{aligned} \tag{78}
$$

This remainder formula allows us to estimate correctly the errors when the hyperdistribution expansion is truncated.

We observe that the Rodriguez expansion for a Gaussian, given in section 4 is also a Taylor expansion i.e. (setting $y = x + \Delta, z = x$)

$$
\delta_\lambda(x + \Delta) = \sum_{n=0}^{\infty} \Delta^n \nabla^n \delta_\lambda(x)/n! \tag{79}
$$

Multiplying both sides by a good (test) function $\phi(x)$, and integrating , we obtain

$$\int_{-\infty}^{+\infty} \delta_\lambda(x+\Delta)\phi(x)dx = \sum_{n=0}^{\infty} \Delta^n \int_{-\infty}^{+\infty} \nabla^n \delta_\lambda(x)\phi(x)dx/n!$$
$$= \sum_{n=0}^{\infty}(-\Delta)^n \int_{-\infty}^{+\infty} \delta_\lambda(x) \nabla^n \phi(x)dx/n! \quad (80)$$

Taking the limit $\lambda \to 0$ of both sides we conclude for the Taylor expansion of the test function i.e.

$$\phi(-\Delta) = \sum_{n=0}^{\infty}(-\Delta)^n \nabla^n \phi(0)/n! \quad (81)$$

Therefore the test function must be real analytic to conclude that

$$\delta(x+\Delta) = e^{\delta\nabla}\delta(x) = \sum_{n=0}^{\infty} \Delta^n \nabla^n \delta(x)/n! \quad (82)$$

To show that caution musat be exercised in treating hyperdistributions as ordinary distributions we point out an example of a function which is smooth but not real-analytic.

$$\phi(x) = e^{-(x^2+1/x^2)} \quad (83)$$

The function ϕ is infinitely differentiable (C^∞) and tapers exponentially at infinity. Nevertheless it has zero derivatives at the origin and it is therefore not real-analytic. As a consequence, the familiar Taylor expansion of the δ function is, strictly speaking, a hyperdistribution.

7 Figure Captions

- Fig 1 Schematic relation among the sections of the paper.

- Fig 2 The Ogura function (solid line) is plotted together with the 5 terms (dashed line) and 9 terms (dot-dashed line) Rodriguez expansion to the Ogura function. Note improved convergence to the Ogura function is seen in the 9 term expansion. In both expansions, the parameter λ is chosen to be 1.0.

- Fig 3 The Ogura function and 5 term and 9 term Rodriguez expansions as in Fig 1, except with $\lambda = 0.75$. Note that compared to the $\lambda = 1.0$ case, deviations between the Ogura function and the Rodriguez expansions are significantly larger.

- Fig 4 One-sided moment plots of $\mu_n = \int_0^x (x')^n f(x')dx'$ for the Ogura function (solid line) and its 5 term Rodtriguez expansion (dashed line) with $\lambda = 1.0$. Note that the μ_n 's for the Ogura function and the Rotriguez expansion coverage as x increases for n=0,2, and 4 while $\mu_6(x)$ for the 5 term Rodriguez expansion differs from the Ogura function plot of $\mu_6(x)$

- Fig 5 One-sided moment plots, as in Fig 3, for the 5 term Rodriguez expansion of the Ogura function with $\lambda = 0.75$. Note that the one-sided moments exhibit larger variations before converging to their larger x values.

- Fig 6 Five term Hermite expansion (dashed line) to the Ogura function (solid line). While convergence to the Ogura function is good over the domain in which the Ogura function exhibits significant values, outside this domain rapid oscillations ("whipping tails") are exhibited.

- Fig 7 One-sided moment expansions of the 5 term Hermite approximation (dashed line) to the Ogura function (solid line). Note that all of the μ_n 's computed for the Hermite expansion deviate from the μ_n 's computed for the Ogura function as x increases, indicating the failure of the Hermite expansion as a global approximation to the Ogura function.

- Fig 8 The extended monopole shown with three values of the radius parameter λ. The graph shows that as $\lambda \rightarrow 0$, we obtain a point source.
- Fig 9 The extended dipole
- Fig 10 The extended quadrupole
- Fig 11 Least square error for the Rodriguez expansion of a gaussian
- Fig 12 least square error og Fig 11 in expanded scale
- Fig 13 the rapidly varying function $f(x) = cosKx\frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}\lambda}$ with $K = 10, \lambda = 3$
- Fig 14 Expanded scale corresponding to Fig 13
- Fig 15 The rapidly varying function with $K = 4, \lambda = 3$
- Fig 16 The least square error for Fig 15
- Fig 17 Reconstruction of a T smeared with a gaussian filter.
- Fig 18 Reconstruction as in Fig 17 with double smearing.
- Fig 19 Simulation of the Grossberg-Todororic neural network with hyperdistributions.

References

[1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions.* Dover Publications Inc., N.Y. (1970).

[2] L. Andrews. *Special Functions for Engineers and Applied Mathematics.* Macmillan Publishing Company, N.Y. (1985).

[3] J.D. Bjorken and S.D. Drell. *Relativistic Quantum Mechanics.* Mc-Graw Hill, N.Y. (1964).

[4] S. Bochner and W.Martin. *Several Complex Variables.* Princeton University Press, Princeton (1948)

[5] E.Butkov. *Mathematical Physics, p.524.* Addison-Wesley Pub. Co. , Reading, MA. (1968).

[6] S. Chapman and T. Cowling. *The Mathematical theory of non- uniform gases; an account of the kinetic theory of viscosity, thermal conduction, and diffusion in gases.* Second Edition, Cambridge University Press, Cambridge (1952).

[7] P.A.M. Dirac, *The Principles of Quantum Mechanics*, Clarendon Press, Oxford (1947).

[8] *Encyclopedic Dictionary of Mathematics, 2nd Ed.* K.Ito,ed., MIT Press, Cambridge, MA (1987).

[9] A.Erdelyi. *Operational Calculus and Generalized Functions.* Holt, Rinehart, and Winston, N.Y. (1962).

[10] W.Feller, *An Introduction to Probability Theory and Its applications,* Wiley, New York (1966).

[11] F.G. Friedlander. *Introduction to the theory of distributions* . Cambridge University Press, N.Y. (1982).

[12] I.M. Gelfand et al. *Generalized Functions.* Acamedic Press, N.Y. (1964-68).

[13] R.C. Gonzalez and P.Wintz, *Digital Image Processing*. Addison- Wesley Publishing Co., Inc., Reading, MA (1987).

[14] M. Greenberg. *Applications of Green 's Function in Science and Engineering*. Prentice-Hall, Inc. N.J. (1971).

[15] S. Grossberg. *Human Vision and Neural Computation*, University Lecture, Boston University, 1989.

[16] I. Halperin. *Introduction to the theory of distributions* Canadian Mathematical Congress, University of Toronto Press, Toronto (1952).

[17] S. Hawking. *A Brief History of Time, p.139*. Bantam Books (1988).

[18] Hes. Helgason. *Groups and Geometric Analysis*, Academic Press, Inc. (1984).

[19] E. Hille and R. Philips. *Functional Analysis and Semi-groups* . American Mathematical Society, (1957).

[20] R.G. Hohfeld, and G.V.H. Sandri, "Remainder Theorem for Hyperdistributions", in preparation (1990).

[21] L. Hormander. *The Analysis of Linear Partial Differential Operators, vol.1, p.181, Paley-Wiener-Schwartz theorem*. Springer-Verlag, Berlin , N.Y. (1983).

[22] M.J. Lighthill, *Fourier Analysis and Generalized Functions*. Cambridge University Press, Cambridge (1970).

[23] J.P. Marchand. *Distributions, p.13*. North Holland Publishing Company, Amsterdam (1962).

[24] A.S. Monin, and A.M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*, (ed. J.L. Lumley), MIT Press, Cambridge (1975).

[25] P. Morse. *Thermal Physics*. Second Edition, W.A. Benjamin , N.Y. (1969).

[26] P.M. Morse and H. Feshbach. *Methods of Theoretical Physics*. McGraw-Hill, N.Y. (1953).

[27] A. Oppenheim, A. Willsky with I. Young. *Signals and Systems* . Englewood Cliffs, N.J. Pretice-Hall. (1983).

[28] W.K. Pratt. *Digital Image Processing.* Willey. N.Y. (1978).

[29] F. Riesz and B. Sz-Nagy. *Translation of Lecons d' Analyse Fonctionelle.* N.Y. Unger, (1955).

[30] G.V.H. Sandri, R.G. Hohlfeld, T.W. Drueding, D. Konstantopoulos, J.I.F. King, and P. Thomas, "Theory and Applications of Generalized Distributions", submitted to Applied Optics. (1989).

[31] L. Schwartz. *Theorie des distributions.* Hermann et Cie., Paris. (1951).

[32] G. Temple, *Generalized Functions.* Proc. Roy. Soc. A, 228, 175-90 (1955).

[33] G. Wentzel. *Quantum Theory of Fields. Translation of Einfuhrung in die Quantentheorie der Wellenfelder. p.177.* Interscience Publishers, N.Y. (1949).

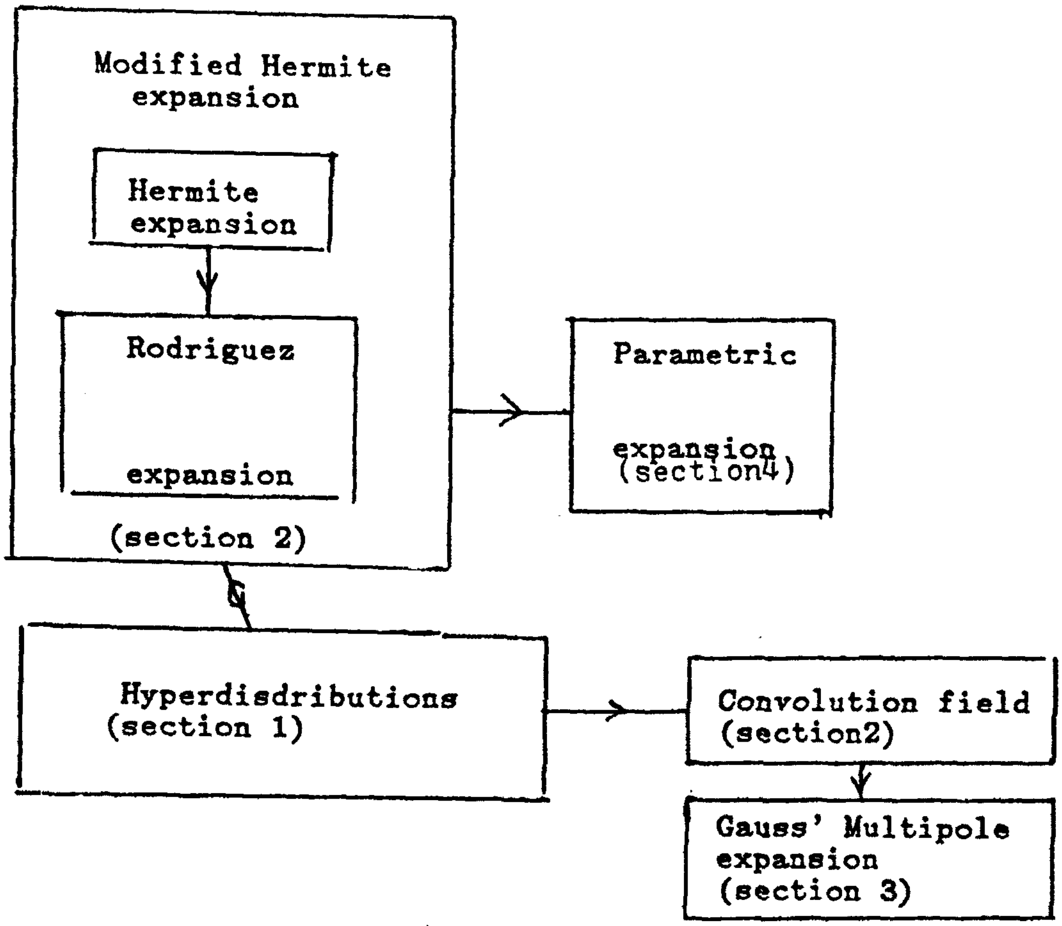
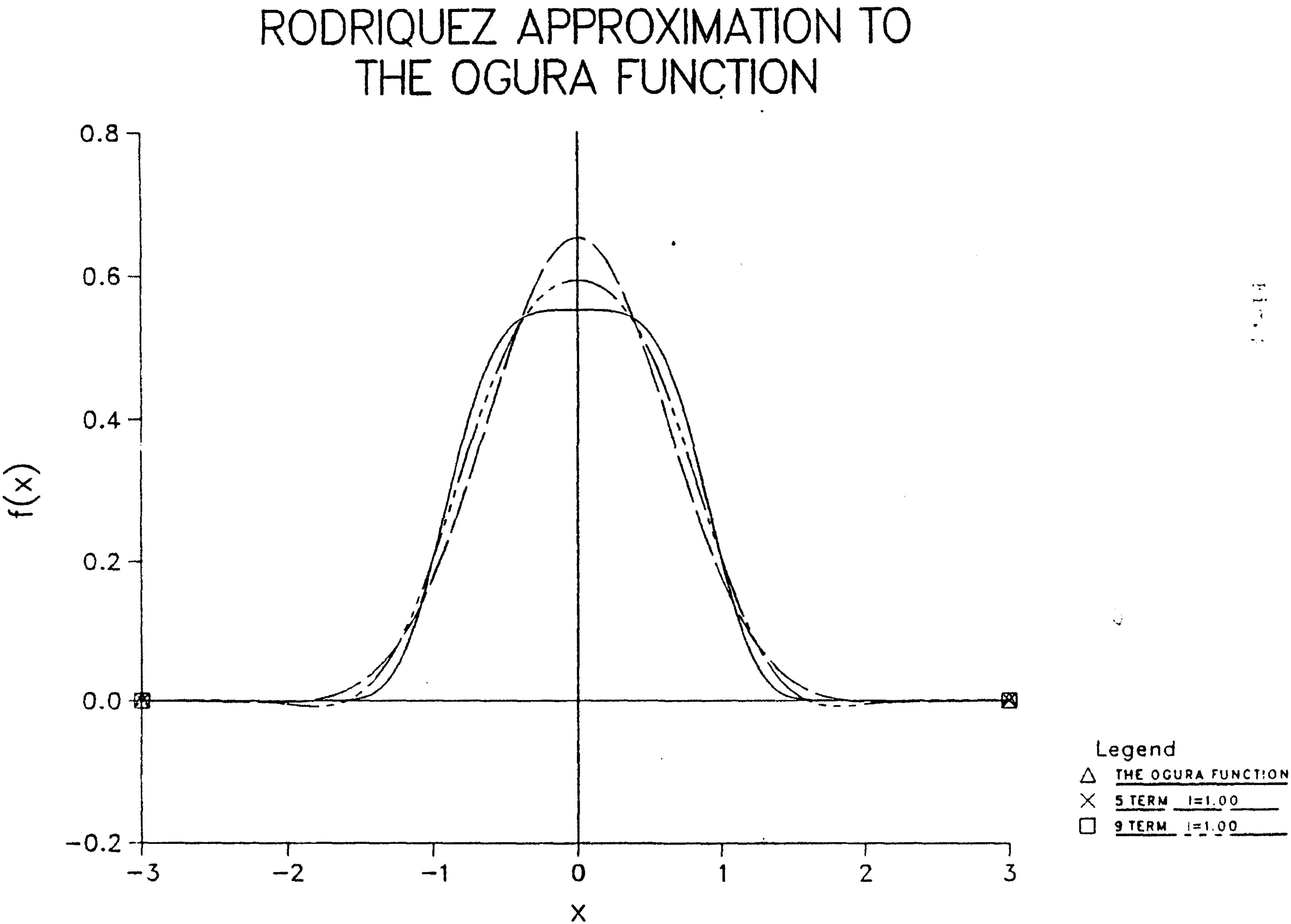
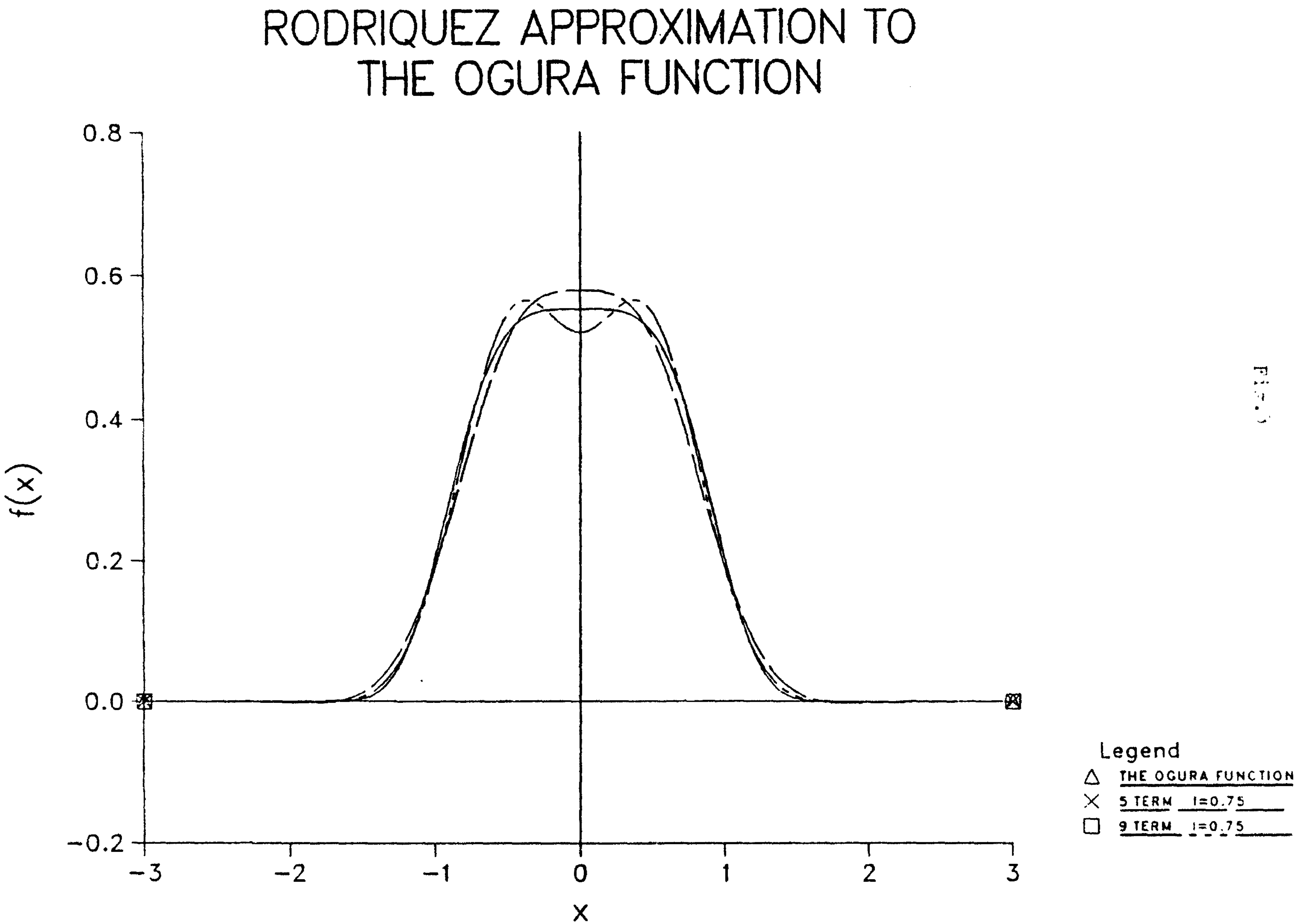
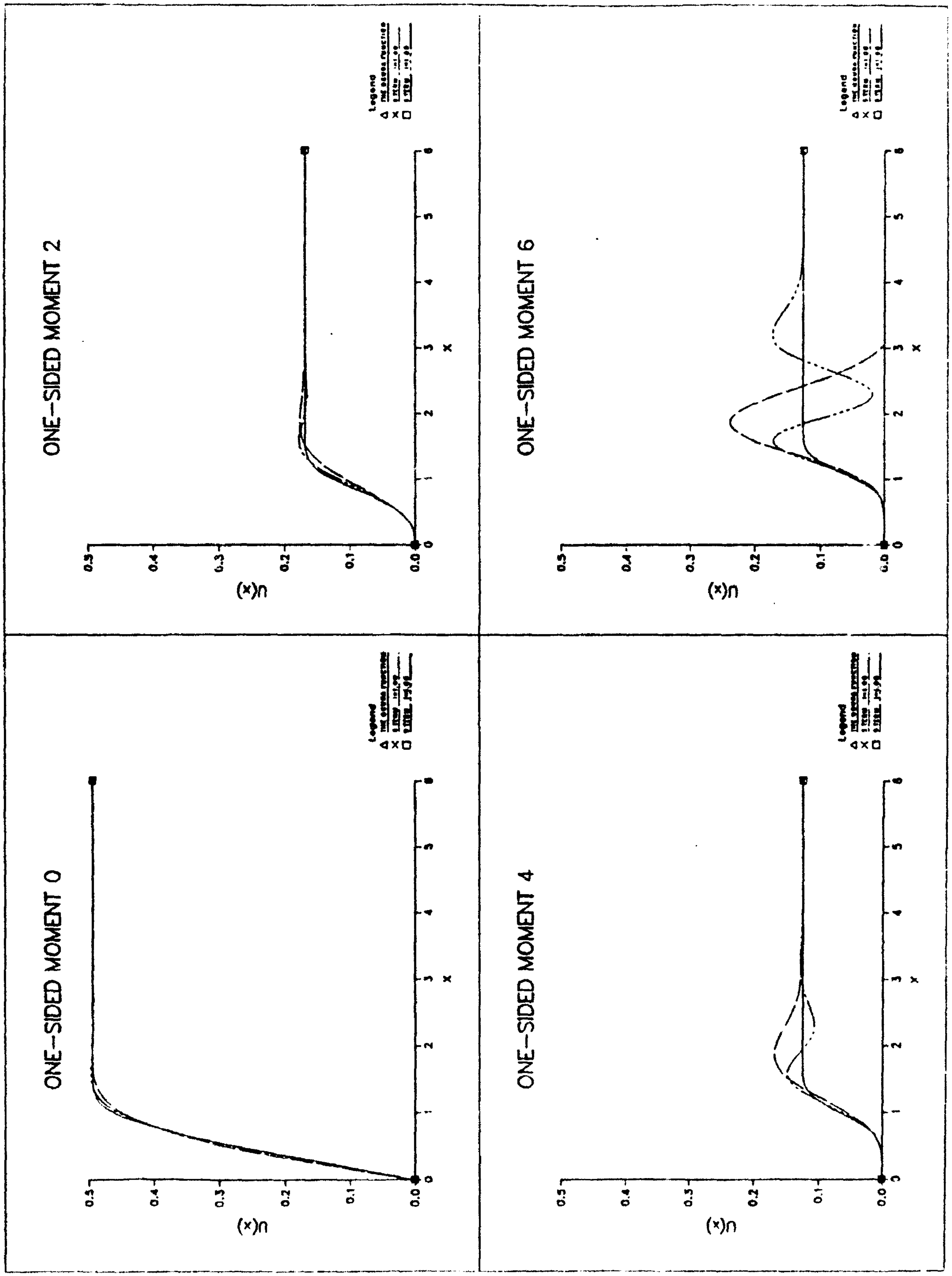
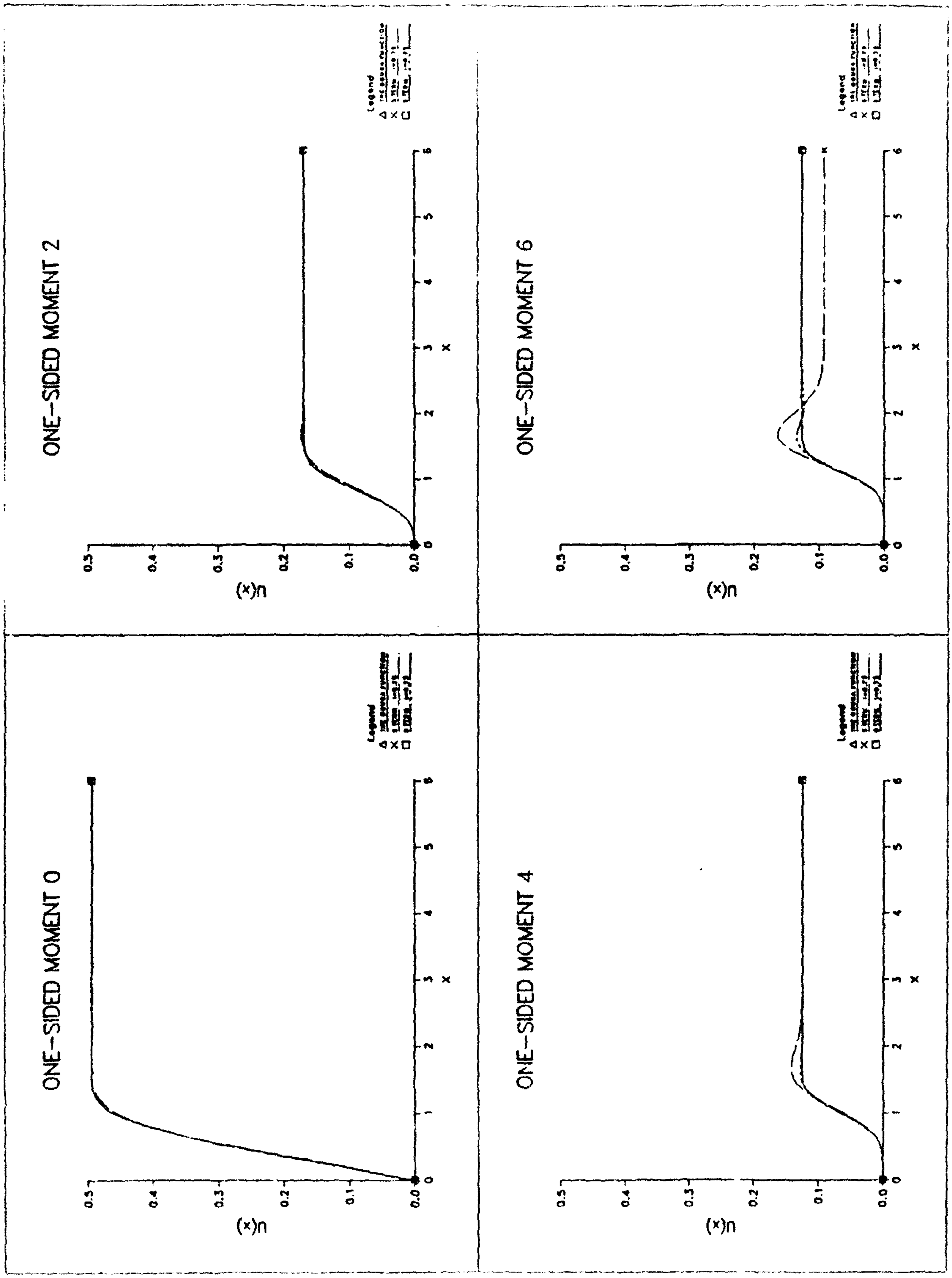


Figure 1.Relations among the sections of this paper.


RODRIQUEZ APPROXIMATION TO
THE OGURA FUNCTION
0.8
0.6
0.4
0.2
0.0
-0.2
f(x)
-3
-2
-1
0
1
2
3
X
Legend
THE OGURA FUNCTION
5 TERM I=1.00
9 TERM I=1.00


RODRIQUEZ APPROXIMATION TO
THE OGURA FUNCTION
0.8
0.6
0.4
0.2
0.0
−0.2
f(x)
−3
−2
−1
0
1
2
3
X
Legend
THE OGURA FUNCTION
5 TERM I=0.75
9 TERM I=0.75


ONE-SIDED MOMENT 2
ONE-SIDED MOMENT 6
ONE-SIDED MOMENT 0
ONE-SIDED MOMENT 4
u(x)
x
Legend


ONE-SIDED MOMENT 2
ONE-SIDED MOMENT 6
ONE-SIDED MOMENT 0
ONE-SIDED MOMENT 4
u(x)
x
Legend

HERMITE APPROXIMATION TO THE OGURA FUNCTION

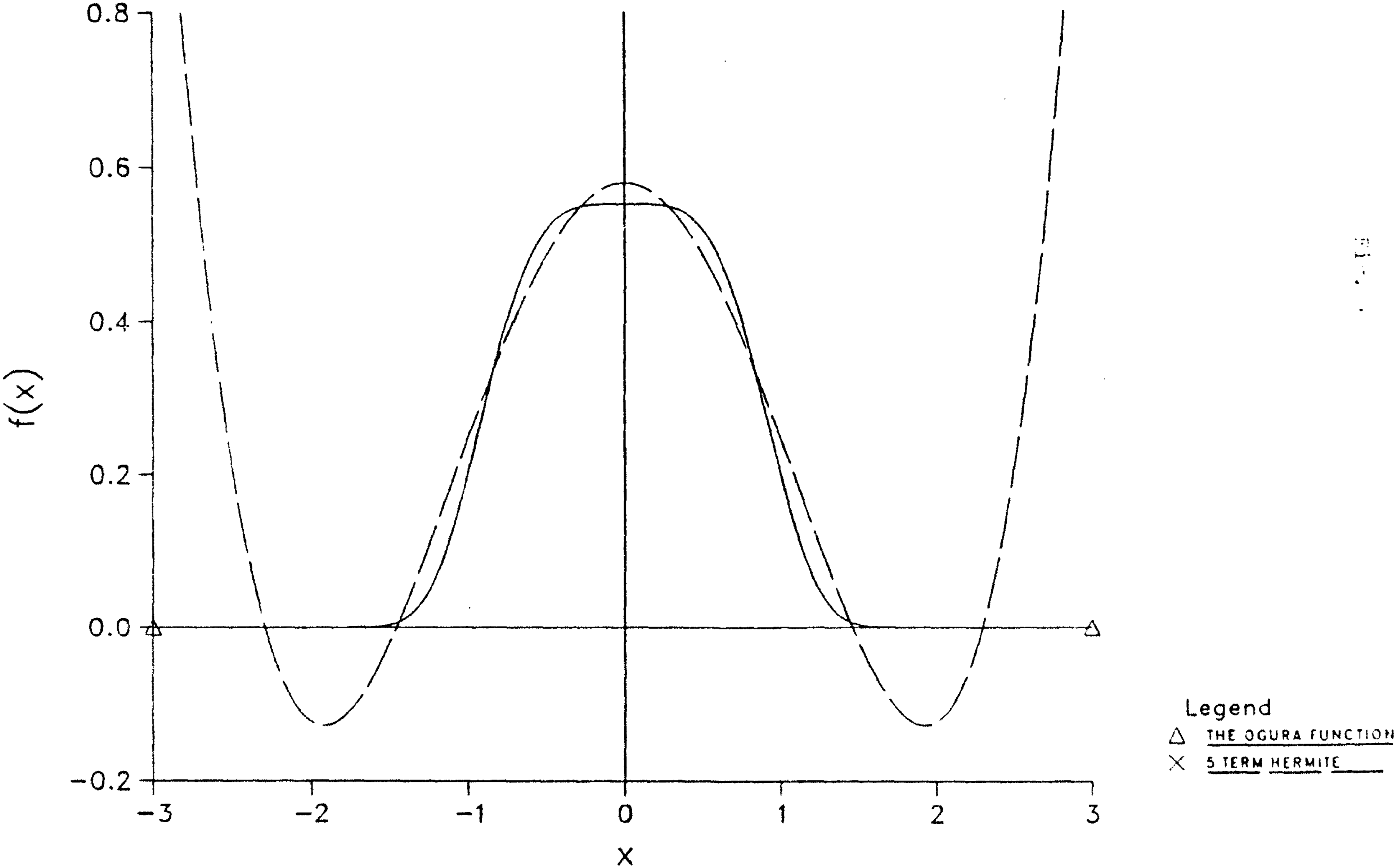
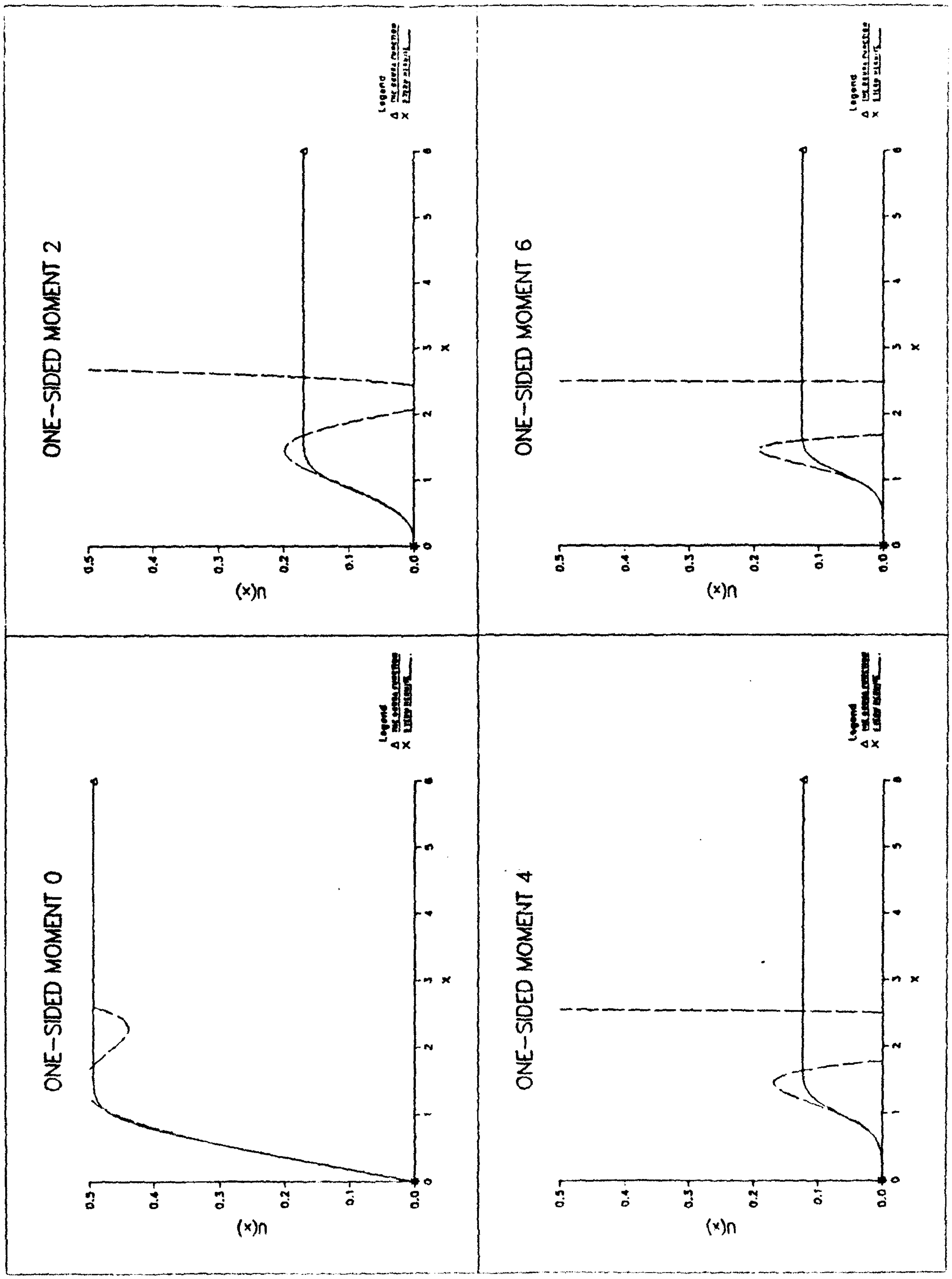
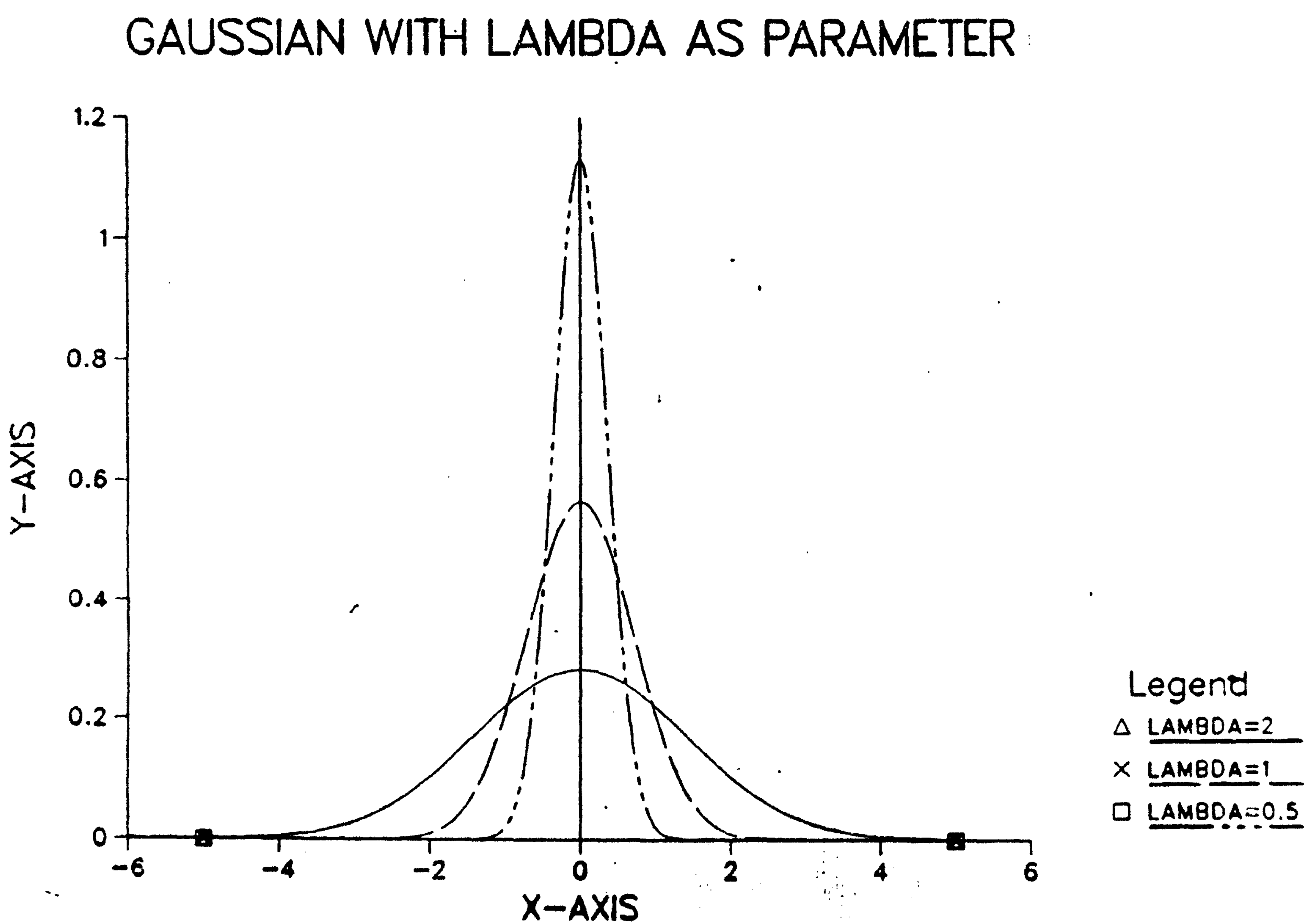
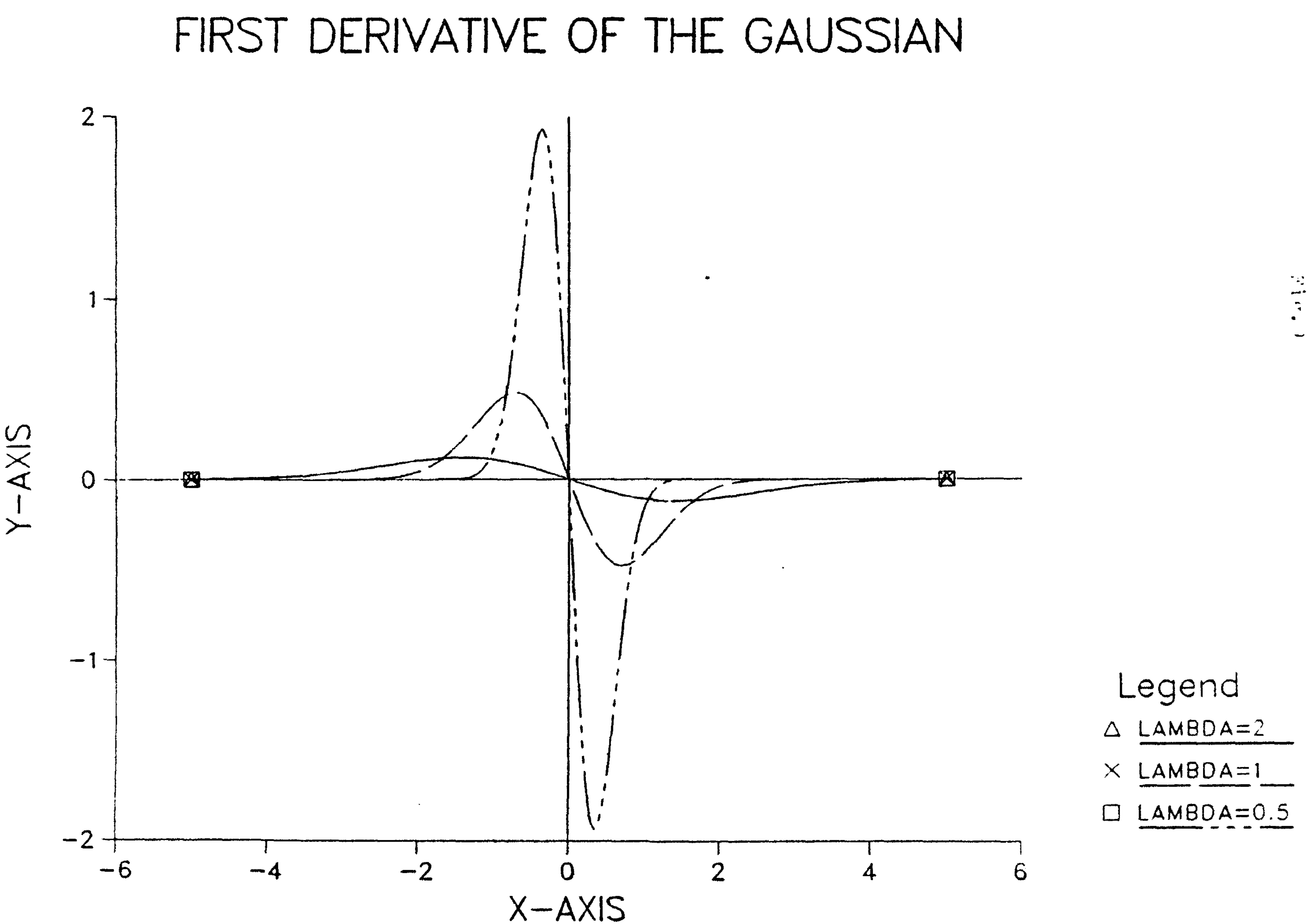


Fig. [illegible]


ONE-SIDED MOMENT 0
ONE-SIDED MOMENT 2
ONE-SIDED MOMENT 4
ONE-SIDED MOMENT 6
u(x)
x
Legend
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6


GAUSSIAN WITH LAMBDA AS PARAMETER
1.2
1
0.8
0.6
0.4
0.2
0
Y-AXIS
-6
-4
-2
0
2
4
6
X-AXIS
Legend
△ LAMBDA=2
× LAMBDA=1
□ LAMBDA=0.5



FIRST DERIVATIVE OF THE GAUSSIAN
2
1
0
−1
−2
Y−AXIS
−6
−4
−2
0
2
4
6
X−AXIS
Legend
LAMBDA=2
LAMBDA=1
LAMBDA=0.5

SECOND DERIVATIVES OF THE GAUSSIAN

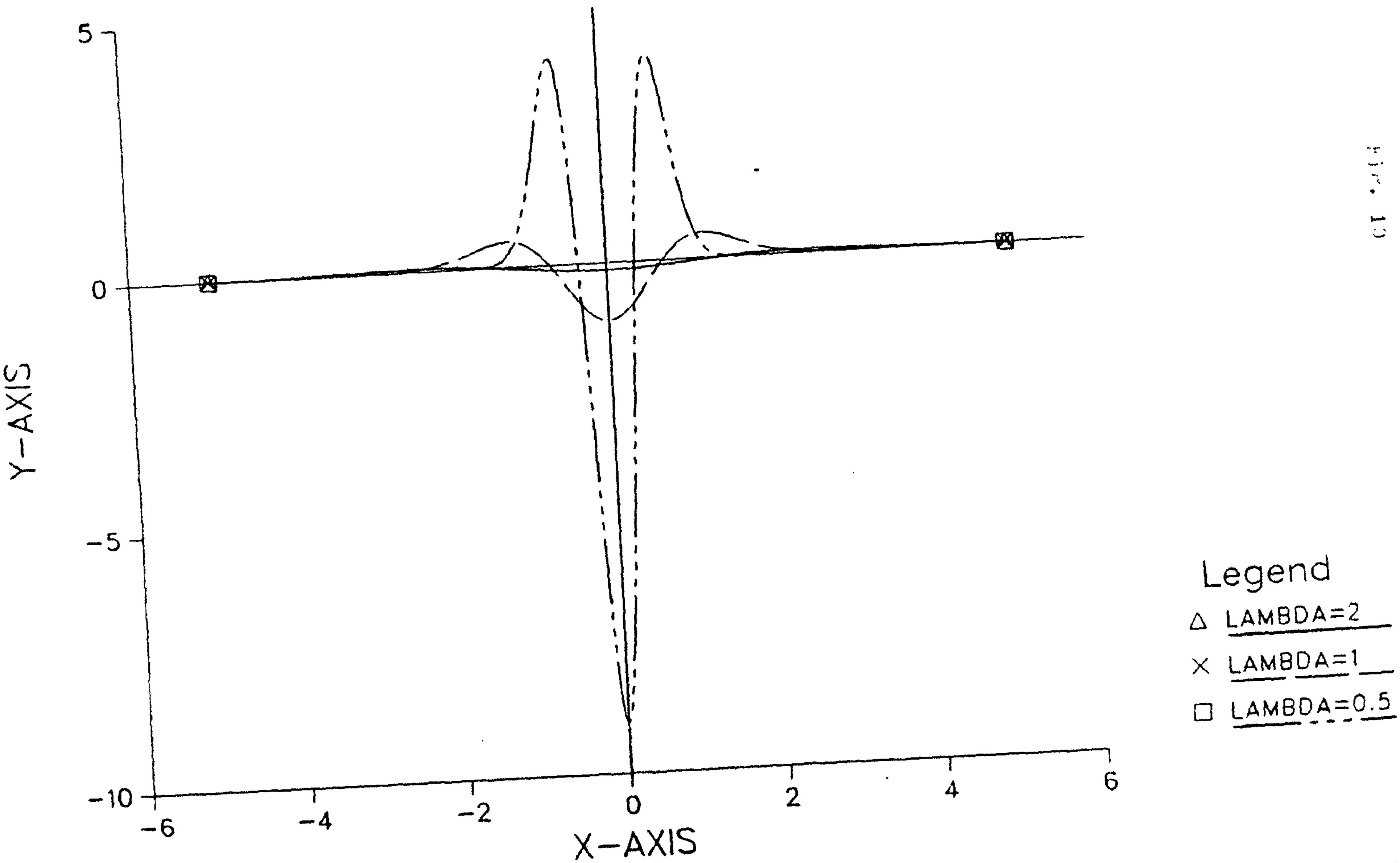
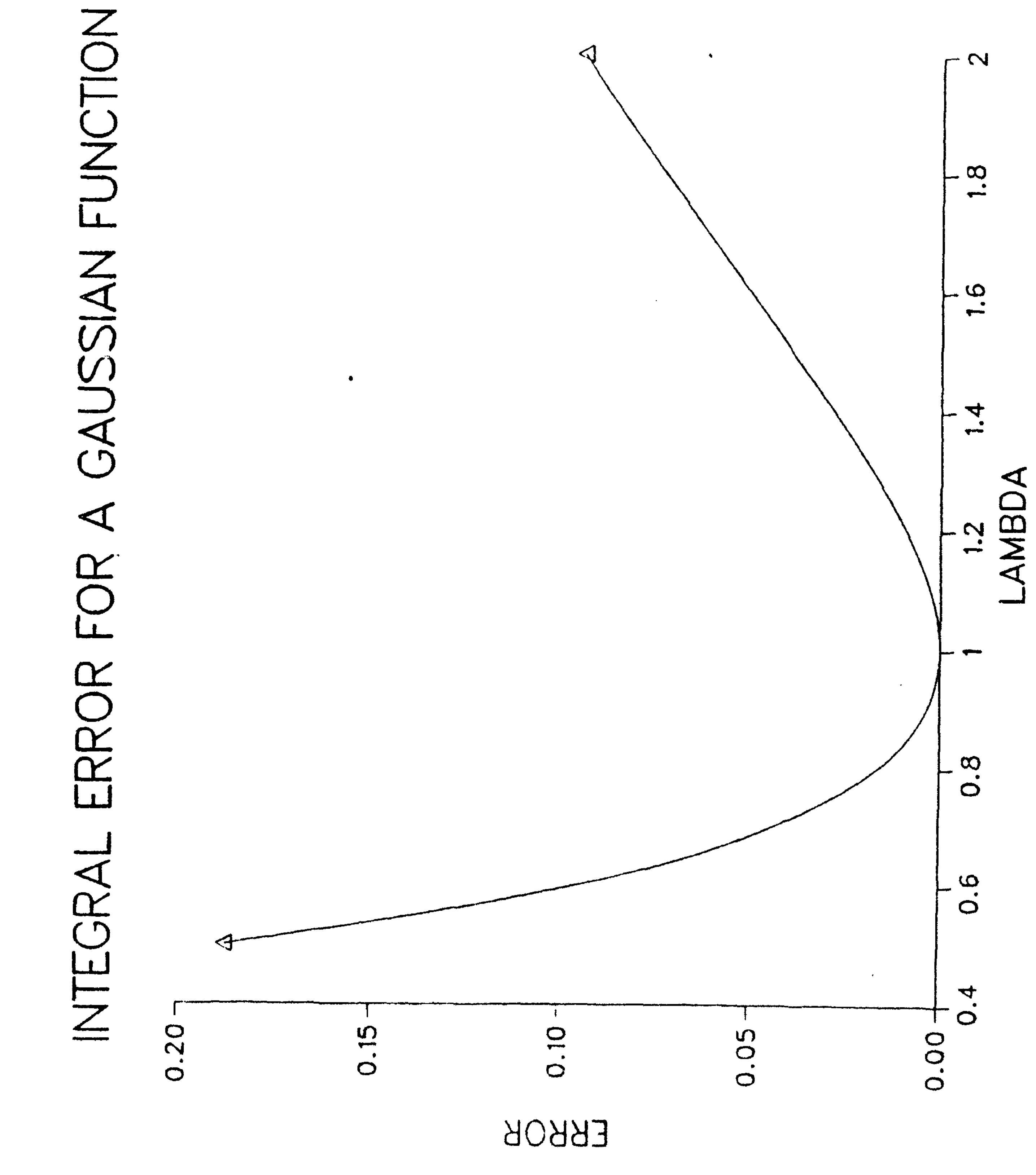


FIG. 10

Fig. 11


INTEGRAL ERROR FOR A GAUSSIAN FUNCTION
ERROR
0.20
0.15
0.10
0.05
0.00
LAMBDA
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Fi. 12

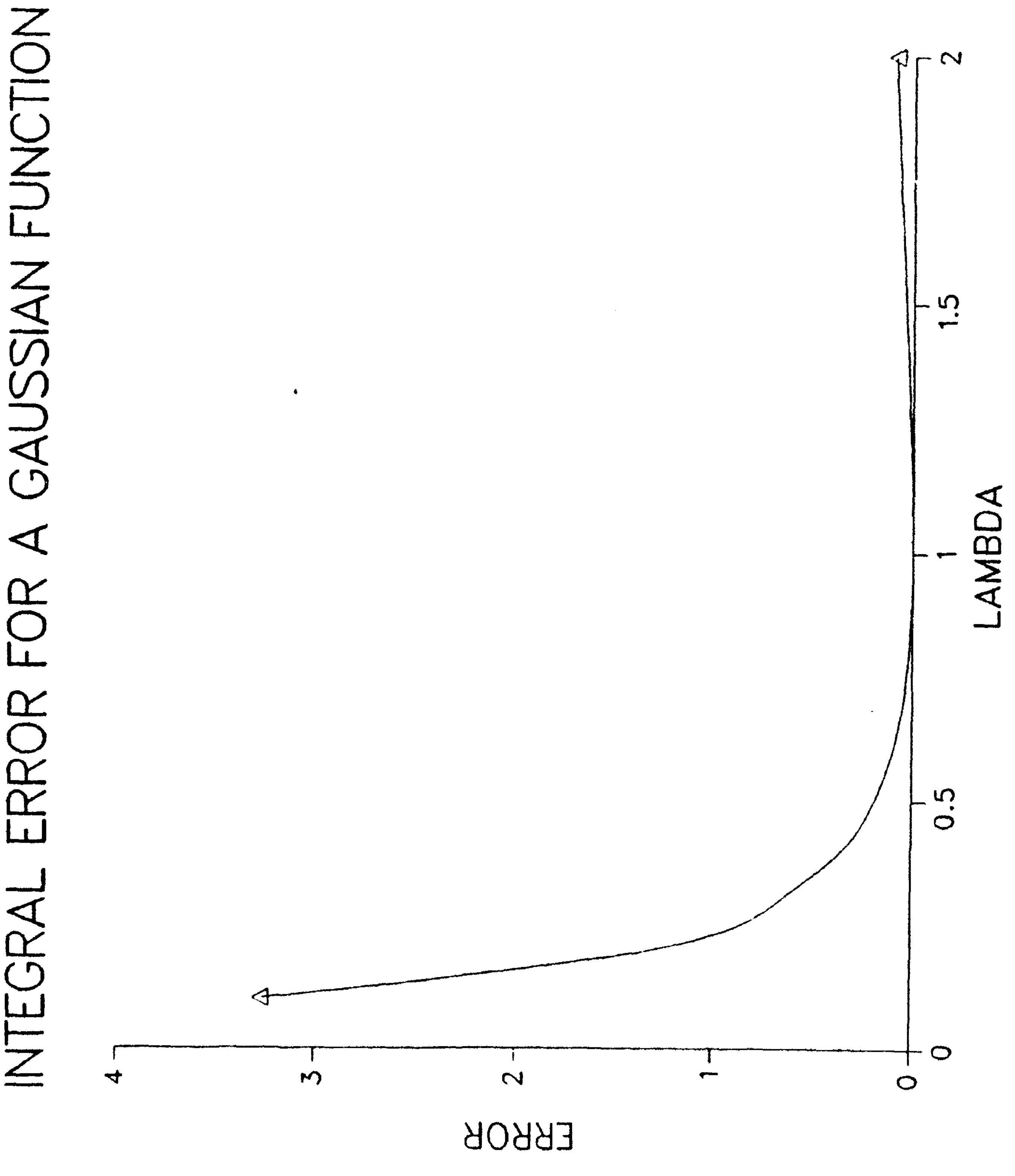
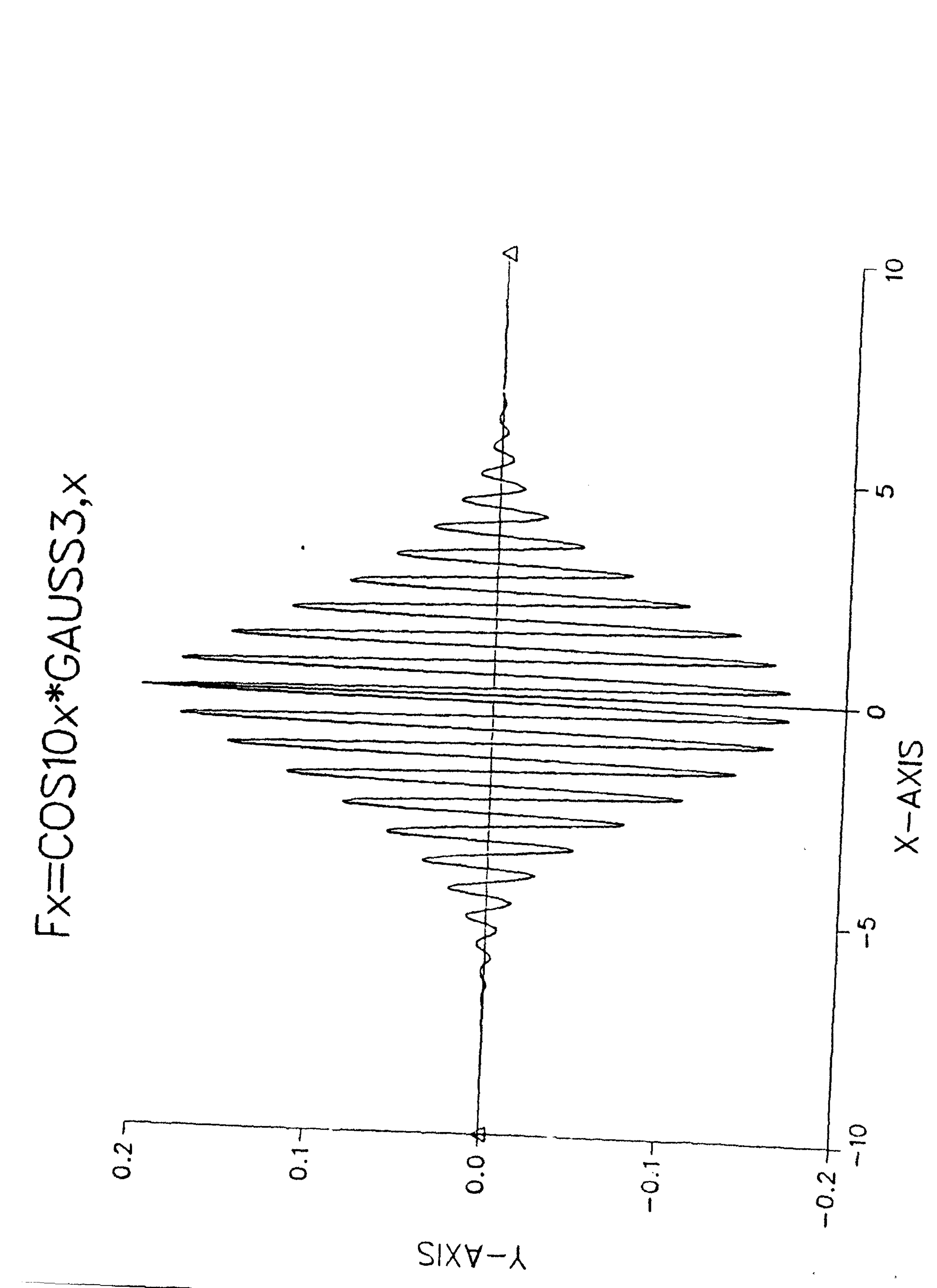
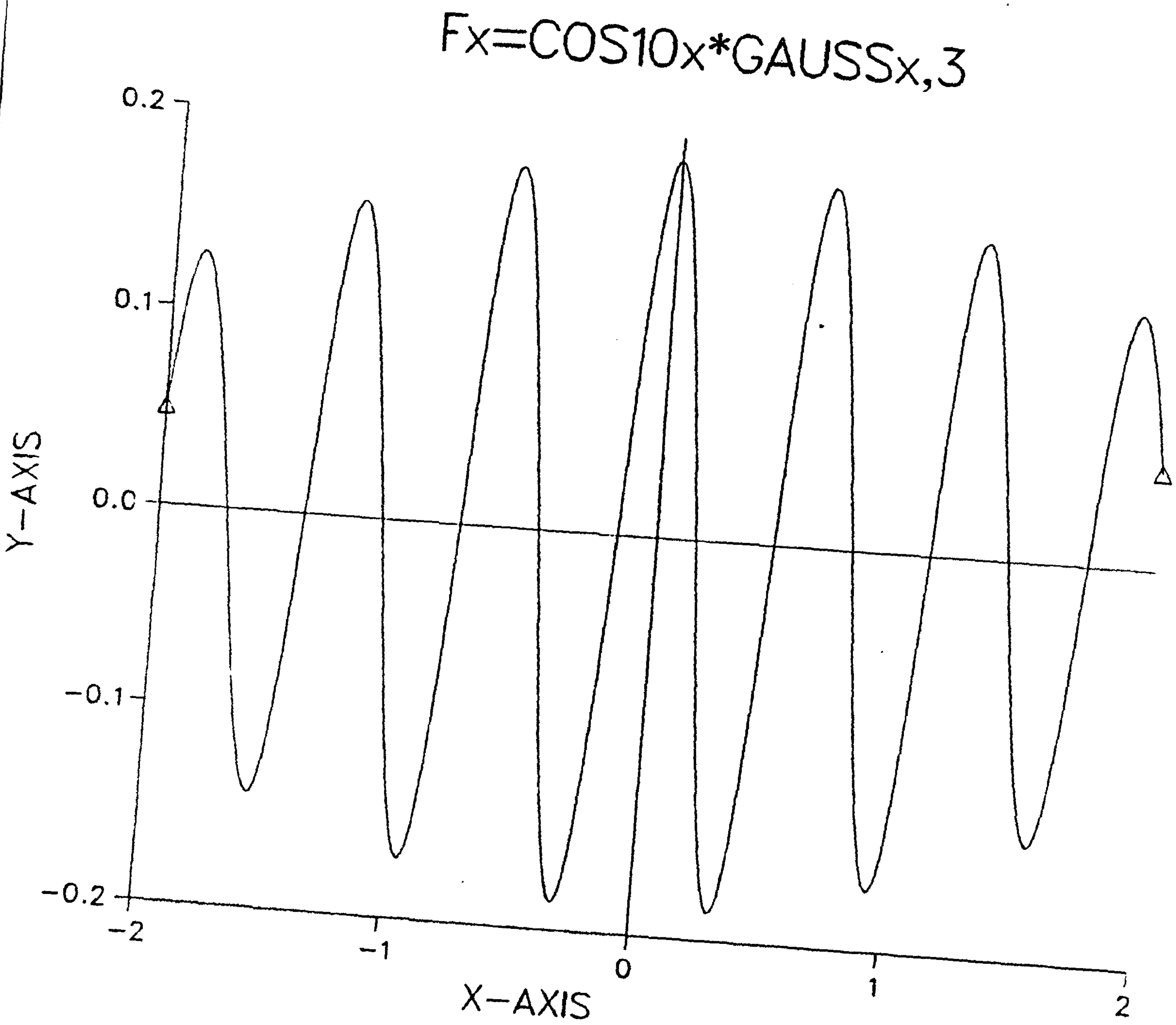
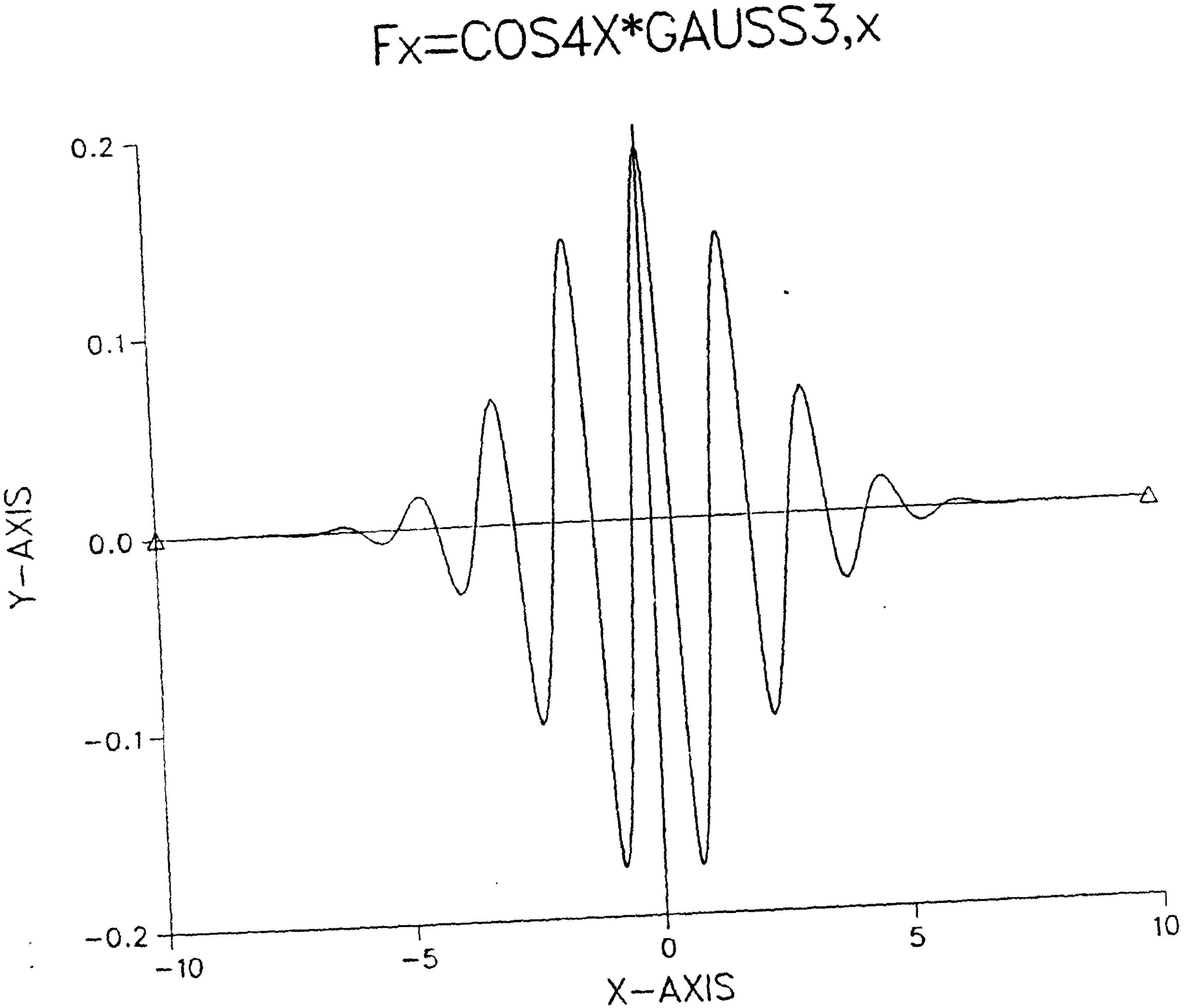
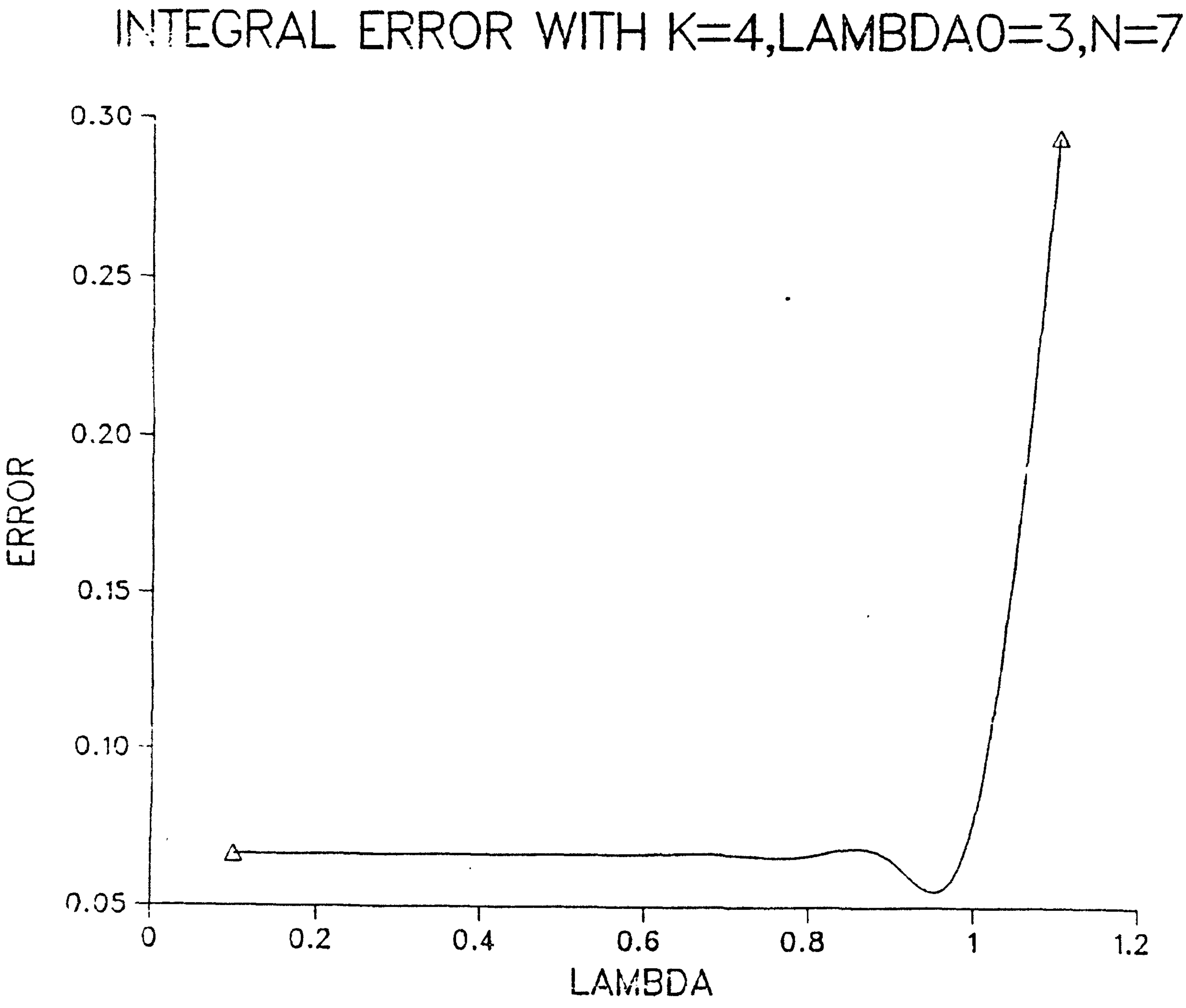

INTEGRAL ERROR FOR A GAUSSIAN FUNCTION
ERROR
0
1
2
3
4
LAMBDA
0
0.5
1
1.5
2

Fig.13




Fx=COS10x*GAUSSx,3
0.2
0.1
0.0
-0.1
-0.2
Y-AXIS
-2
-1
0
1
2
X-AXIS


Fx=COS4X*GAUSS3,x
0.2
0.1
0.0
−0.1
−0.2
Y−AXIS
−10
−5
0
5
10
X−AXIS



INTEGRAL ERROR WITH K=4,LAMBDA0=3,N=7
ERROR
0.30
0.25
0.20
0.15
0.10
0.05
0
0.2
0.4
0.6
0.8
1
1.2
LAMBDA

Hyperdistributions II: Two Dimensional Analysis

T. Drueding[1], T. Fang[2], R. Hohlfeld[1], W. Ma[1], G. Sandri[1]

[1] Boston University College of Engineering
110 Cummingtion Street, Boston, MA 02215

[2] Massachusetts Technological Laboratory, Inc.
330 Pleasant Street, Belmont, MA 02178

To Be Submitted for Publication: SIAM Journal for Numerical Analysis

Hyperdistribution II: Two Dimensional Analysis

We continue in this paper the development of a new technique for the calculation of convolution products and their inverses. We focus in this paper on two- dimensional problems. This is accomplished by constructing a class of singular "functions" of two variables , hyperdistributions of two variables, that form a closed algebraic field with the convolution product as the multiplicative operation. In this paper we consider functions of two variables. This "two dimensional" algebra can be applied to image processing. Furthermore, in this paper we use the construction of hyperdistributions to obtain a novel parametric approximation method which is the two-dimensional analogue of the one- dimensional expansion developed in our previous paper. We demonstrate the use of our approximation method with simple examples.

Introduction

We continue to develop a novel tool for the calculation of convolution products and their inverses. We denote these new mathematical objects by "hyperdistributions" or "generalised functions" . These are singular "functions", which we construct to form an algebraic abelian field with the convolution product as the multiplicative operation.

Signal processing and analysis are natural applications of hyperdistributions in one (one dimensional) variable. Image processing is a corresponding application for hyperdistributions in a two dimensional variable. Tomography corresponds to a three dimensional variable and the budding field of space-time processing corresponds to four dimensions.

The outline of the present paper parallels the previous paper is as follows: in section I we introduce hyperdistributions in two dimensions heuristically; we discuss the convolution group and derive a remainder theorem. In section II we construct rigorously hyperdistributions by a modified Hermite polynomial expansion in two dimensions and we use the tools of the Christoffel-Darboux theory to obtain sufficient conditions for L^2 convergence. In section III we show that Gauss' multipole expansion in two dimensions is obtained explicitly as a simple application of the hyperdistribution inverse. Finally, in section IV we expand a two dimensional gaussian function in terms of derivatives of a different gaussian to demonstrate the use of our new parametric expansion and the concurrent minimization of error.

We note that, in effect we achieve a method for establishing and approximating solutions of integral equations of convolution form in two variables. We demonstrate with examples that there are cases for which our method is applicable, but Fourier transform methods fail. Comparing hyperdistributions with Fourier transforms, our method requires the calculation of the moments of the given functions and of the kernel rather than calculation of their Fourier coefficients. This property motivates consideration of examples for which our method is preferable to Fourier transform techniques. The applications given are motivated by image deconvolution and by the analysis of turbulent flows.

1 Heuristic definition of Hyperdistribution

We introduce a general approximation whose integral properties are the focus of interest. Our approximation displays in configuration space the properties of the classical moment generating expansion for the Fourier transform of the probability distribution. Moments, and even shapes, are shown to be captured well by our expansion. In addition, our expansion allows consideration of "functions" which are more singular than temperate (i.e. Fourier-analyzable) distributions, but that can be represented by infinite sequences of distributions. We call these "functions" generalised distributions. We show that generalised distributions form the appropriate framework for carrying out the process of deconvolution. in fact, we give a straightforward algorithm , the "Bochner-Martin algebra", to compute explicitly the convolution inverse of any generalised distribution. Applications of the method to simple optical deconvolutions are shown to be straightforward. These applications have been motivated originally by our studies of optics and turbulent flows. We have since become aware of the much wider applicability of our methods.

1.1 Taylor and moment expansions

The theory of generalised distributions is built on ideas related to Dirac's delta function, which is technically a "generalised function". The Dirac delta function, $\delta(x)$, has the properties

$$\begin{aligned} \int\int_{-\infty}^{+\infty} \delta(x,y)dxdy &= \int\int_{-\infty}^{+\infty} \delta(x)\delta(y)dxdy \\ &= (\int_{-\infty}^{+\infty} \delta(x)dx)^2 = 1 \end{aligned} \tag{1}$$

and

$$f(x,y) = \int\int_{-\infty}^{+\infty} dx'dy' f(x',y')\delta(x-x',y-y') \tag{2}$$

for any suitably smooth function, $f(x)$. We term equation (2), "Dirac 's identity".

The Dirac delta function is symmetric in its argument, i.e. $\delta(x-x',y-y') = \delta(x'-x,y'-y)$ and, since the Dirac delta function is a generalised function, we may Taylor expand $\delta(x'-x,y'-y)$ about (x',y'),

$$\delta(x'-x,y'-y) = \delta(x',y') - x'\nabla_{x'}\delta(x',y') - y'\nabla_{y'}\delta(x',y') + \ldots + R^{n+1} \tag{3}$$

This expression and similar ones below are assumed to hold under integration. Equation (3) allows one to compute a "local" approximation to $f(x,y)$, since if we substitute this expansion into Dirac's identity, we recover the usual Taylor expansion of $f(x,y)$ about $(x,y) = (0,0)$.

$$f(x,y) = f(0,0) + x\nabla_x f(0,0) + y\nabla_y f(0,0) + \ldots + R_T^{n+1}[f] \qquad (4)$$

This approximation is local in the sense that it requires derivatives of $f(x)$ at a single point $x = 0$, and in general has a limited radius of convergence. On the other hand, if we expand $\delta(x - x', y - y')$ about x, we have

$$\delta(x - x', y - y') = \delta(x,y) - x'\nabla_x\delta(x,y) - y'\nabla_y\delta(x,y) + \ldots + R^{n+1} \qquad (5)$$

When this series is substituted into Dirac's identity, we obtain

$$f(x,y) = M^0\delta(x)\delta(y) - M^1\nabla\delta(x)\delta(y) + \ldots + R_M^{n+1}[f] \qquad (6)$$

The coefficients M^n, defined by

$$\begin{aligned} M^0 &= \int\!\!\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy \\ M'_x &= \int\!\!\int_{-\infty}^{+\infty} x f(x,y)\,dx\,dy \\ M'_y &= \int\!\!\int_{-\infty}^{+\infty} y f(x,y)\,dx\,dy \end{aligned} \qquad (7)$$

are simply the moments of the function $f(x,y)$. Therefore, equation (6) is an approximation of $f(x,y)$ involving global information about $f(x,y)$, that is, the moments of the function.

This then may be taken as a motivation for our definition of a generalised distribution as a function which may be written in the form,

$$\begin{aligned} f(x,y) &= \sum_{n=0}^{\infty} a_n \odot \nabla^n\delta(x)\delta(y) \\ a_0 \odot \nabla^0 &= a_0 \\ a_1 \odot \nabla^1 &= a_{1x}\nabla_x + a_{1y}\nabla_y \\ a_2 \odot \nabla^2 &= a_{2xx}\nabla_x^2 + 2a_{2xy}\nabla_x\nabla_y + a_{2yy}\nabla_y^2 \end{aligned} \qquad (8)$$

The a_n values are coefficients given by

$$a_n = (-1)^n M^n / n! \qquad (9)$$

Note that equation (8) is equivalent to the familiar moment generating expansion of probability theory (see equation 6).

1.2 The Convolution Group

If we have two generalised distributions f_1 and f_2, their linear combination $\lambda f_1 + \mu f_2$ is also a generalised distribution (where λ and μ are real coefficients). The pth derivative of a generalised distribution, $\nabla^p f(x,y)$, is a generalised distribution. Also , the convolution of two generalised distribution $f_1 * f_2$ is a generalised distribution. These may all be thought of as "closure properties" of generalised distributions.

Generalised distributions allow us to make effective computation of the convolution inverse. Given a generalised distribution f, the desired convolution inverse $In[f]$ satifies

$$f * In[f] = \delta(x,y) \tag{10}$$

Here $\delta(x,y)$ represents Dirac's delta function, which is the identity of the convolution operation. We shall show by our construction that $In[f]$ is a generalised distribution. Writing the convolution explicitly,

$$f * In[f] = \int\int_{-\infty}^{+\infty} dx'dy' f(x',y') In[f](x-x', y-y') = \delta(x)\delta(y) \tag{11}$$

It is now necessary to compute the product of the sums and match coefficients. Taking $f(x,y) = \sum_{n=0}^{\infty} a_n \odot \nabla^n \delta(x)\delta(y)$,and $g(x,y) = \sum_{n=0}^{\infty} b_n \odot \nabla^n \delta(x)\delta(y)$, we see that the computation of the convolution inverse is effectively the determination of a collection of b_n values, given a set of a_n values. Substituting into equation (11),

$$\nabla^n \delta(x,y) * \nabla^m \delta(x,y) = \nabla^{n+m} \delta(x,y) \tag{12}$$

This gives

$$\sum_p (a_p \sum_q b_q \nabla^{p+q} \delta(x,y)) = \delta(x,y) \tag{13}$$

or equivalently with $r = p + q$

$$\sum_{p=0}^{\infty} \sum_{r=0}^{\infty} a_p b_{r-p} \nabla^r \delta(x,y) = \delta(x,y) \tag{14}$$

Matching coefficients on the left and right hand sides implies that only the $r = 0$ term survives. The result is a linear system of equations for the b

values in terms of the a values. It is easier to see the behavior by writing the first equations in this linear system.

$$\begin{aligned} a_{00}b_{00} &= 1 \\ a_{00}b_{10} + a_{10}b_{00} &= 0 \\ a_{00}b_{01} + a_{01}b_{00} &= 0 \end{aligned} \tag{15}$$

and so forth. Thus, we can see the computation of the terms of the convolution inverse.

1.3 Fourier Transforms and Simple Examples

Lastly, we shall want to consider the Fourier transform of a generalised distribution. Again taking $f(x,y) = \sum_{n=0}^{\infty} a_n \nabla^n \delta(x,y)$, we can immmediately evaluate the Fourier transform as:

$$\int\int_{-\infty}^{+\infty} dx dy f(x,y) e^{ik_1x+ik_2y} = \sum_{n=0}^{\infty} a_n \odot (-ik)^n \tag{16}$$

where for the last step we have used $\nabla^n e^{ik_1x+ik_2y} = (ik)^n$. The Fourier transform of a generalised distribution may seen to be a power series.

The Fourier transform of a function is the "moment generating function" because of equation (9). Thus the basic requirement for the validity of writing a function as a generalised distribution is that tis moments be finite, or more stringently, that its Fourier transform be real analytic.

Simple explicit examples of convolution inverses are obtained from the standard Green's functions. The two-dimensional Helmholtz equation, for example, can be treated equally easily with generalised distributions and with Fourier transforms to obtain both the Green's function and its convolution inverse. Thus , we have

$$(\nabla^2 + 1/4)e^{-\frac{|x|+|y|}{2}} = -\delta(x)\delta(y) \tag{17}$$

Application of equation (15) shows that only b_0 and b_2 contribute to the convolution inverse

$$In|e^{-\frac{|x|+|y|}{2}}| = \delta(x,y)/2 - \delta''(x,y) \tag{18}$$

This result is easily reproduced by using Fourier transforms and is easily verified using equation (10).

In contrast, the convolution inverse of the Gaussian function cannot be obtained as a Fourier transform, but it is easily obtained using generalised distributions. It may be verified that for a Gaussian function of width σ, the convolution inverse is readily obtained from equation (15), while calculation using Fourier transforms leads to a divergent result. Calculation of momnets and use equation (15) yields

$$\begin{aligned} a_{2n} &= (\sigma/\sqrt{2})^n/n! \\ b_{2n} &= (-1)^n(\sigma/\sqrt{2})^n/n! \end{aligned} \tag{19}$$

2 Rigorous Construction of Hyperdistributions

In this section we first modify the classical expansion in the Hermite polynomials. This expansion is "superior" to a power series expansion in that the terms are orthogonal, making the error orthogonal to the approximation. Our modification maintains this advantage. Furthuremore, the standard Christoffel -Darboux analysis gives a very useful sufficient condition for convergence that our expansion inherits from the Hermite polynomial expansion. Since our expansion utilises the Rodriguez formula for the Hermite polynomials, we call it the " Rodriguez expansion".

2.1 Modified Hermite Expansion: Rodriguez expansion

In effect, we discuss here a systematic pointwise approximation of generalised distributions. This approximation method is analogous to the method developed by Temple to approximate distributions by smooth functions. The Temple method has become widely known through Lighthill's monograph: Fourier Series and Generalised Functions.

Generalised distributions are approximate by a modification of the classical expansion in Hermite polynomials (hereafter called the "Hermite expansion") which is suggested by the Rodriguez formula:

$$(-1)^{n+m} H_n(x) H_m(y) e^{-(x^2+y^2)} = \nabla_x^n \nabla_y^m e^{-(x^2+y^2)} \tag{20}$$

To see how the Hermite expansion can be transformed into an approximation for generalised distributions, consider a function $f(x,y)$ which we want to represent as a generalised distribution. Multiply $f(x,y)$ by

$$\frac{e^{-\frac{x^2+y^2}{\lambda^2}}}{\pi\lambda} \tag{21}$$

then expand the resulting expression in terms of the "scaled Hermite polynomials"

$$H_n^\lambda(x) = \lambda^{-n} H_n(x/\lambda) \tag{22}$$

whose definition is justified by the fromula (25) below. We obtain the expression

$$f(x,y)\frac{e^{-\frac{x^2+y^2}{\lambda^2}}}{\pi\lambda^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a^{\lambda}_{n,m} H^{\lambda}_n(x) H^{\lambda}_m(y)\frac{e^{-\frac{x^2+y^2}{\lambda^2}}}{\pi\lambda^2} \tag{23}$$

Multiply both sides of the equation by the normalised Gaussian introduced by Temple, i.e.

$$\delta_\lambda(x,y) = \frac{e^{-\frac{x^2+y^2}{\lambda^2}}}{\pi\lambda^2} \tag{24}$$

And, observe that by rescaling the Rodriguez formula (20) we can write

$$(-1)^{n+m} H^{\lambda}_n(x) H^{\lambda}_m(y)\delta_\lambda(x,y) = \nabla^n_x \nabla^m_y \delta_\lambda(x,y) \tag{25}$$

The final result is

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c^{\lambda}_{n,m} \nabla^m_x \nabla^m_y \delta_\lambda(x,y) \tag{26}$$

where we have introduced the coefficients

$$\begin{aligned} c^{\lambda}_{n,m} &= (-1)^{n+m}\alpha^{\lambda}_{n,m} \\ \alpha^{\lambda}_{n,m} &= \frac{\lambda^{2n+2m}}{2^{n+m} n! m!}\int\int_{-\infty}^{+\infty} f(x,y) H_n(x) H_m(y)\,dx\,dy \end{aligned} \tag{27}$$

When we let λ tend to zero, we have a representation of $f(x)$ as a generalised distribution. We call the expansion (26) the "Rodriguez expansion for $f(x)$ with width λ". The width parameter is a novel feature of the Rodriguez expansion when compared to standard expansions in complete sets of basis functions. The standard expansions do not contain a free parameter. The advantages of the Rodriguez expansion will be demonstrated below in the context of the theory of generalised distributions and of convolution inverses.

We consider as an example a function familiar from the analysis of turbulence spectra, the "Ogura" function

$$f(x,y) = Ne^{-(x^2+y^4)} \tag{28}$$

With N determined by the normalisation condition,

$$\int\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1 \tag{29}$$

we have

$$N = (1/4)\Gamma^2(1/4) = (1.8128)^2 \tag{30}$$

The Hermite expansion exhibits pointwise convergence, albeit with "whipping tails". By contrast, the Rodriguez expansion exhibits pointwise convergence without "whipping tails". As a consequence, the Rodriguez expansion also yields a global approximation by accurately representing the moments. Furthurmore, the Rodriguez approximations has an adjustable parameter which may be selected to optimise the rate of convergence. The Christoffel-Darboux theory gives the necessary condition for convergence in appropriate L^2. For the Hermite expansion, we have

$$\int\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} f^2(x,y)dxdy < \infty \tag{31}$$

Inserting the transformation that leads to the Rodriguez expansion, we find

$$\int\int_{-\infty}^{+\infty} e^{x^2+y^2} f^2(x,y)dxdy < \infty \tag{32}$$

for the corresponding L^2 definition for the Rodriguez series.

2.2 Definition of Hyperdistributions

In order to describe the process of antidiffusion, we have introduced a class of highly singular "functions", that are precisely defined in this section. Our process for defining hyperdistributions parallels the Temple definition (generalised function) as a good sequence of good functions. Good functions are smooth and tapered. More precisely, they are total point functions that are differentiable to all orders (C^∞), and decay at $\pm\infty$ faster that any power. Good functions play the role of "testing" a sequence of good functions for weak convergence. In fact, a sequence of good functions is a distribution if

$$\lim_{n\to\infty} \int\int_{-\infty}^{+\infty} \phi(x,y) f_n(x,y)dxdy < \infty \tag{33}$$

for all good functions ϕ.

Since we conceive of hyperdistributions as "generalised" distributions, we are in fact implementing a second order generalisation of functions. Consequently, we need a double test as a convergence criterion. We implement

this criterion by introducing *very good* functions $G_\Lambda(x,y)$ with the following properties:

1. $G_\Lambda(x,y)$ is smooth, that is, differentiable to all orders, C^∞.
2. $G_\Lambda(x,y)$ is essentially compact, i.e. it has a gaussian decay at $\pm\infty$: $G_\Lambda(x,y) \approx Ne^{-(x^2+y^2)}$.

We will assume for convenience that G_Λ is normalised to unity:

$$\int\int_{-\infty}^{+\infty} G_\Lambda(x,y)dxdy = 1 \tag{34}$$

We define the width of G_Λ by

$$\Lambda^2/4 = \int_{-\infty}^{+\infty} (x-\bar{x})^2(y-\bar{y})^2 G_\Lambda(x,y)dxdy \tag{35}$$

A primary example of very good function is the gaussian, which we denote by $\delta_\lambda(x,y)$:

$$\delta_\lambda(x,y) = \frac{e^{-\frac{(x^2+y^2)}{\lambda^2}}}{\pi\lambda^2} \tag{36}$$

We now introduce a sequence of very good functions defined by

$$\mathcal{H}^\lambda_\backslash(\S,\dagger) = \sum_{||=\prime}^{\backslash} \dashv_{||} \otimes \nabla^{||}\delta_\lambda(\S,\dagger) \tag{37}$$

The sequence $\{\mathcal{H}^\lambda_\backslash\}_{\lambda,\backslash}$, where *lambda* is a nonnegative real and n is a natural number, is a good sequence if, for all good functions ϕ and for all very good functions G_Λ, there exits a Λ_0 such that, for all $\Lambda > \Lambda_0$,

$$\lim_{\substack{n\to\infty \\ \lambda\to 0}} \int\int_{-\infty}^{+\infty} \phi(x,y)(\mathcal{H}^\lambda_\backslash * \mathcal{G}_*)\lceil\S\lceil\dagger < \infty \tag{38}$$

We note that $e^{\pm t\nabla^2}\delta(x,y)$ are hyperdistributions. The sum (hyperdistribution)

$$\sum_{k=0}^{\infty} a_k \otimes \nabla^k\delta(x,y) \tag{39}$$

can thus be viewed either as a sequence of "good" distributions as $\lambda \to 0$:

$$\sum_{k=0}^{n} a_k \odot \nabla^k \delta(x, y) \tag{40}$$

or, as $n \to \infty$, as sequence of good functions:

$$\sum_{k=0}^{\infty} a_k \odot \nabla^k \delta_\lambda(x, y) \tag{41}$$

The latter representation is a *Rodriguez expansion.* The Rodriguez formula for the Hermite polynomials can be used to show the derivatives of a gaussian form a complete set of orthogonal polynomials in an L^2 space. And thus the Rodriguez expansion yields a very useful point function approximation to any hyperdistribution:

$$\sum_{k=0}^{\infty} a_{kl}(-1)^{k+l} H_k(x/\lambda) H_l(y/\lambda) \delta_\lambda(x, y)/\lambda^{k+l} \tag{42}$$

where $H_n(x)$ denotes the Hermite Polynomial in x of order n.

3 The Multipole Expansion

We start with the familiar Poisson equation of potential theory,

$$\nabla^2\phi = \rho \tag{43}$$

This equation is rewritten with the help of the (infinite domain) Green's function

$$\begin{aligned} G(x,y) &= ln(r)/2/\pi \\ \nabla^2 G &= \delta(x,y) \\ r^2 &= x^2 + y^2 \end{aligned} \tag{44}$$

We can then rewrite the "potential", ϕ, in terms of the "charged distribution", ρ as

$$\phi = G * \rho \tag{45}$$

Introduce Q with the property

$$Q * \rho = \delta \tag{46}$$

Convolving both sides of eq(45) with Q and using the commutative and associative properties of the * product, we find

$$\begin{aligned} Q * \phi &= Q * (G * \rho) \\ &= (Q * \rho) * G \\ &= G \end{aligned} \tag{47}$$

Solve eq(43) for ϕ in terms of the given G by computing the convolution inverse,

$$\phi(x,y) = \sum_{k=0}^{\infty} \lambda^k \odot \nabla^k G(x,y) \tag{48}$$

which is Gauss' multipole series with coefficients

$$\lambda_{kl} = \frac{(-1)^{k+l}}{k!l!} \int\int r^{\otimes n} \rho(x,y)\,dx\,dy \tag{49}$$

where $\otimes$ denotes tensor product and $x^{\otimes k}$ is the tensor power of the 3-vector x.

Substituting (19) into (18) we find a familiar expression

$$\phi(x,y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\int\int_{-\infty}^{+\infty} r^{\otimes k} \rho(x,y) dx dy \right) \otimes \nabla^k \frac{ln(r)}{2\pi} \qquad (50)$$

which is a standard result in potential theory.

We can interpret the Rodriguez expansion as a generalised multipole expansion that includes a size or radius paramter, λ. Thus our monopole generates the point source, which is a Dirac delta-function as a gaussian of width λ. We recover the standard multipoles when $\lambda \to 0$. Our expansion has the form

$$f(x,y) = a_0^\lambda \delta_\lambda(x,y) + (a_{1x}^\lambda \nabla_x + a_{1y}^\lambda \nabla_y)\delta_\lambda(x,y) + \ldots \qquad (51)$$

The ingredients of the expansion are the basis functions

$$\delta_\lambda, \nabla_x\delta_\lambda, \nabla_y\delta_\lambda, \ldots \qquad (52)$$

and the coefficients

$$a_0^\lambda, \qquad a_{1x}^\lambda, \qquad a_{1y}^\lambda, \ldots \qquad (53)$$

all of which depend on the size parameter λ. The basis functions are, explicitly

$$\delta_\lambda(x,y) = \frac{e^{-\frac{(x^2+y^2)}{\lambda^2}}}{\pi\lambda^2} \qquad (54)$$

$$\nabla_x\delta_\lambda(x,y) = -2x\frac{e^{-\frac{(x^2+y^2)}{\lambda^2}}}{\pi\lambda^2} \qquad (55)$$

$$\nabla_y\delta_\lambda(x,y) = -2y\frac{e^{-\frac{(x^2+y^2)}{\lambda^2}}}{\pi\lambda^2} \qquad (56)$$

The corresponding expansion coefficients are given by

$$a_0^\lambda \approx \int\int_{-\infty}^{+\infty} f(x,y) dx dy, \qquad (\lambda \to 0) \qquad (57)$$

This coefficient represents the total area, or alternatively, the total mass or charge of the source function. Also

$$a_{1x}^\lambda \approx \int\int_{-\infty}^{+\infty} x f(x,y) dx dy, \qquad (\lambda \to 0) \qquad (58)$$

which represents the net dipole of the source. In mechanical terms a^{λ}_{1x} defines the center of mass of the source function. Finally

$$a^{\lambda}_{2xx} \approx \int\int_{-\infty}^{+\infty} x^2 f(x,y)dxdy, \qquad (\lambda \to 0) \tag{59}$$

which represents the quadrupole of the source distributions. In mechanical terms this is the moment of inertia of the mass distribution. Higher order basis function and coefficients have analogous interpretations.

4 Expansion of a Function in Rodriguez Series and its Optimization

Our expansion preserves the classical properties that are derivable for orthogonal polynomial expansions (as opposed to power series), but also adds an important new feature: the size or radius parameter that generates the Gaussian picture of point monopole, dipole, quadrupole, etc., to a scenario in which we can allow for extended sources. We find from the convergence condition derived from the Christoffel Darboux theory in section II that convergence holds generally on a semi-infinite range of λ. This remarkable freedom is exploited in this section to optimize the rate of convergence of the expansion. This optimization results in determining the value of λ for which a minimum number of terms is determined in order to obtain a given tolerance. We measure the tolerance, as usual , by the least square fit integrated over the entire function. We define the error

$$\epsilon(N,\lambda) = \int\int_{-\infty}^{+\infty} dxdy(f(x,y) - \sum_{k=0}^{N} a_k^\lambda \odot \nabla^k \delta)^2 \tag{60}$$

We apply the formula to a standard gaussian, i.e. we take

$$f(x,y) = e^{-(x^2+y^2)}/\pi \tag{61}$$

we then find, after some calculations

$$a_{2k}^\lambda = \sum_{k=0}^{n-1} \frac{(-1)^k 2^{k-2n}(2n-2k-1)!!}{k!(2n-2k)!} + (-1)^n 2^{-2n}/n! \tag{62}$$

where the double factorial contains only odd terms. We can now express the standard gaussian in the Rodriguez form:

$$\frac{e^{-(x^2+y^2)}}{\pi} = \sum_{n=0}^{\infty} \frac{(1-\lambda^2)^n}{2^{2n} n!} \nabla^{2n} \delta_\lambda(x,y) \tag{63}$$

A proof based on useful linear operator relations is as follows. We can check by differentiation in t and x that

$$\frac{e^{-\frac{(x^2+y^2)}{4t}}}{4\pi t} = e^{t\nabla^2}\delta(x,y) \tag{64}$$

We now let

$$4t = \lambda^2 \tag{65}$$

As a consequence, we can write

$$\frac{e^{-(x^2+y^2)/\lambda^2}}{\pi\lambda^2} = e^{\lambda^2\nabla^2/4}\delta(x,y) = \delta_\lambda(x,y) \tag{66}$$

We now consider the identity

$$e^{\nabla^2/4}\delta(x)\delta(y) = e^{\frac{1-\lambda^2}{4}\nabla^2}\delta_\lambda(x,y) \tag{67}$$

expanding the R.H.S.

$$\begin{aligned} e^{\nabla^2/4}\delta(x)\delta(y) &= \frac{e^{-\frac{x^2+y^2}{\lambda^2}}}{\pi\lambda^2} \\ &= \sum_{n=0}^{\infty}\frac{(1-\lambda^2)^n}{2^{2n}n!}\nabla^{2n}\delta_\lambda(x,y) \end{aligned} \tag{68}$$

The method of proof will be used later. Furthurmore, it provides a welcome check on the rather difficult calculations of coefficients. We see by inspection that $\lambda = 1$ is optimal. We also consider the function

$$f(x,y) = cos(kx)cos(ky)\frac{e^{-\frac{x^2+y^2}{\lambda^2}}}{\pi\lambda^2} \tag{69}$$

5 Conclusions

We have seen that generalised distribution provide a method for solving Fredholm integral equations of the convolution type which is competitive with the current methods that employ Fourier transforms. We have also seen that generalised distributions can be approximated numerically by sequences of smooth functions. This procedure is analogous to that of approximating Dirac *delta* functions by sequences of narrowing Gaussian functions. In this second paper we have used two dimensional examples for illustrative purposes . We will discuss applications in higher dimensions and extensions of the theory of generalised distributions separately.

6 Appendix: Remainder Theorem for Hyperdistributions

In this appendix we give the remainder theorem for hyperdistributions. The remainder formula is of great use in calculating Lagrange and Cauchy estimates for the terms neglected. This feature of our expansion, like the presence of the adjustable "radius" parameter, is unique to our expansion and it is not shared by other orthogonal functions expansion.

The general form of the Taylor expansion for two distinct points $\vec{y}$ and $\vec{z}$ is:

$$F(\vec{y}) = F(\vec{z}) + (\vec{y} - \vec{z}) \cdot \vec{\nabla} F(\vec{z}) + \ldots + R^{n+1} \tag{70}$$

$$R^{n+1} = \int_{\vec{z}}^{\vec{y}} d\vec{t}(\vec{y} - \vec{t})^n \otimes \nabla^{n+1} F(\vec{t})/n! \tag{71}$$

The formula for the remainder is in fact an identity which is proven by recursion integration by parts.

We now give the appropriate remainder for the two dual expansions discussed in section 1.

6.1 Taylor (local) expansion

We use

$$\begin{aligned} F(\vec{x}) &= \delta(\vec{x} - \vec{x'}) \\ &= \delta(\vec{x'} - \vec{x}) \\ \vec{y} &= \vec{x'} - \vec{x} \\ \vec{z} &= \vec{x'} \end{aligned} \tag{72}$$

We have

$$\delta(\vec{x'} - \vec{x}) = \delta(\vec{x'}) - \vec{x} \cdot \vec{\nabla}\delta(\vec{x'}) + \int_{\vec{x'}}^{\vec{x'} - \vec{x}} d\vec{t}(\vec{x'} - \vec{x} - \vec{t})^n \otimes \nabla^{n+1}\delta(\vec{t})/n! \tag{73}$$

Then we have, using Dirac's identity

$$\begin{aligned} f(\vec{x}) &= \int_{R^n} \delta(\vec{x} - \vec{x'}) f(\vec{x'}) dx' \\ &= f(\vec{0}) - \vec{x} \cdot \vec{\nabla} f(\vec{0}) + \ldots + R^{n+1} \end{aligned} \tag{74}$$

with the remarkable relation

$$R^{n+1} = \int_{\vec{0}}^{\vec{x}} d\vec{t}(\vec{x} - \vec{t})^n \otimes \nabla^{n+1} f(\vec{t})/n! \tag{75}$$

6.2 Moment (global) expansion

We set

$$\begin{aligned} F(\vec{x}) &= \delta(\vec{x} - \vec{x}') \\ \vec{y} &= \vec{x} - \vec{x}' \\ \vec{z} &= \vec{x} \end{aligned} \tag{76}$$

We then have

$$\delta(\vec{x} - \vec{x}') = \delta(\vec{x}) - \vec{x}' \cdot \vec{\nabla}\delta(\vec{x}) + \ldots + \int_{\vec{x}}^{\vec{x}-\vec{x}'} d\vec{t}(\vec{x} - \vec{x}' - \vec{t})^n \otimes \nabla^n \delta(\vec{t})/n! \tag{77}$$

Using Dirac identity we conclude

$$f(\vec{x}) = \delta(\vec{x}) \int_{R^n} f(\vec{x}')d^n x' + \ldots \tag{78}$$

This remainder formula allows us to estimate correctly the errors when the hyperdistribution expansion is truncated.

We observe that the Rodriguez expansion for a Gaussian, given in section 4 is also a Taylor expansion i.e. (setting $y = x + \Delta, z = x$)

$$\delta_\lambda(\vec{x} + \vec{\Delta}) = \sum_{n=0}^{\infty} \Delta^n \otimes \nabla^n \delta_\lambda(\vec{x})/n! \tag{79}$$

Multiplying both sides by a good (test) function $\phi(x)$, and integrating, we obtain

$$\int_{R^n} \delta_\lambda(\vec{x} + \vec{\Delta})\phi(\vec{x})d\vec{x} = \sum_{n=0}^{\infty}(-\vec{\Delta})^n \otimes \int_{R^n} \delta_\lambda(\vec{x})\nabla^n \phi(\vec{x})d\vec{x}/n! \tag{80}$$

Taking the limit $\lambda \to 0$ of both sides we conclude for the Taylor expansion of the test function i.e.

$$\phi(-\vec{\Delta}) = \sum_{n=0}^{\infty}(-\vec{\Delta})^n \otimes \nabla^n \phi(\vec{0})/n! \tag{81}$$

Therefore the test function must be real analytic to conclude that

$$\delta(\vec{x} + \vec{\Delta}) = e^{\vec{\Delta}\cdot\vec{\nabla}}\delta(\vec{x}) = \sum_{n=0}^{\infty}(\vec{\Delta})^n \cdot \nabla^n \delta(x)/n! \tag{82}$$

To show that caution must be exercised in treating hyperdistributions as ordinary distributions we point out an example of a function which is smooth but not real analytic.

$$\phi(x,y) = e^{-(x^2+y^2+\frac{1}{x^2}+\frac{1}{y^2})} \tag{83}$$

The function ϕ is infinitely differentiable (C^∞) and tapers exponentially at infinity. Nevertheless, it has zero derivatives at the origin and it is therefore not real analytic. As a consequence, the familar Taylor expansion of the δ function is, strictly speaking, a hyperdistribution.